The Equations of Motion

The principle of least action

The principle of least action (Hamilton’s Principle) states that physical systems will move between times \( t_1 \) and \( t_2 \) so as to minimize the integral

\[
S = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt,
\]

where \( S \) is the action and \( L \) is the Lagrangian of the system.

After doing the calculus to minimize the action, we get Lagrange’s Equations:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.
\]

Dynamically, Lagrangians are equivalent up to a total time derivative of a function of coordinates and time. So the Lagrangians \( L(q, q, t) \) and \( L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{df(q, t)}{dt} \) yield the same equations of motion, because the extra term of \( L' \) gives a constant term in the action that vanishes upon minimization. Rescaling by a constant also has no effect upon the motion (arbitrary units).

Galileo’s relativity principle

In an inertial frame, from which space and time are homogeneous and isotropic, the Lagrangian cannot depend explicitly upon a position vector \( \mathbf{r} \) or time \( t \) associated with a particle. Hence the Lagrangian for a free particle of velocity \( \mathbf{v} \) must depend only upon the velocity of the particle. From the isotropic requirement, it also must be independent of direction. Thus the Lagrangian has a functional form \( L = L(v^2) \). From the lack of dependence on \( \mathbf{r} \), Lagrange’s equations give that

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{v}} = 0,
\]

and because \( L \) depends only upon the velocity, that \( \dot{v} = \) a constant. This is the law of inertia (Newton’s first law). Galileo’s relativity principle essentially asserts the invariance of equations of motion under Galilean transformations.

The Lagrangian for a free particle

Applying Galileo’s relativity principle to the Lagrangian \( L \) of a free particle, a transformation \( \mathbf{v} \to \mathbf{v} + \mathbf{e} \) must result in a new Lagrangian differing only by a total time derivative. Expanding \( L(v^2 + 2v \cdot e + e^2) = L' \) in powers of \( e \), we get

\[
L' = L + \frac{\partial L}{\partial v^2} 2v \cdot e.
\]

The second term is a total time derivative iff it is a linear function of \( v \), so \( \partial L/\partial v^2 \) must be independent of the velocity, and so the Lagrangian must be proportional to the square of the velocity:

\[
L = \frac{1}{2} m v^2.
\]

This defines the mass \( m \). The mass cannot be negative, or else the action would have no minimum. The mass is also only really meaningful when the linearity of the Lagrangian is considered, where it is found that the ratios of masses of particles is what matters.

The Lagrangian for a system of particles

For non-interacting particles (indexed by \( a \)) in a closed system, the Lagrangian is of the form

\[
L = \sum a m_a v_a^2.
\]

When interactions are allowed, the Lagrangian containing the kinetic energy must be augmented with a function of the coordinates, the potential energy:

\[
L = \sum a m_a v_a^2 - U(t_1, t_2, \ldots)
\]

Galileo’s relativity principle requires that changes in the motion of the system due to the potential energy are propagated instantly. Substituting this Lagrangian into Lagrange’s equations gives the equations of motion:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial q_a}
\]

\[
m_a \frac{\dot{v}_a}{dt} = -\frac{\partial U}{\partial q_a},
\]

known in this form as Newton’s equations, and

\[
F_a = -\frac{\partial U}{\partial q_a}
\]

is called the force on the \( a \)th particle. From this, it is clear that the accelerations of the particles depend only upon their coordinates.

In generalized, as opposed to Cartesian, coordinates the kinetic energy remains a quadratic function of the velocities, but may also depend upon the coordinates (coordinate dependence is different from position dependence).

When considering systems \( A \) and \( B \) that interact, their combination being closed, the Lagrangian is of the form \( L = T_A(q_A, \dot{q}_A) + T_B(q_B, \dot{q}_B) - U(q_A, q_B) \). This is like \( A \) moving through an external field due to \( B \), and by linearity we get \( L_A = T_A(q_A, q_A) - U(q_A, q_A(t)) \).

For example, an interacting particle has Lagrangian

\[
L = \frac{1}{2} mv^2 - U(r, t)
\]

and equation of motion

\[
m \dot{v} = -\frac{\partial U}{\partial r}.
\]

When the same force acts on the particle at any position in the field, the field is uniform and has a potential energy of the form

\[
U = -F \cdot r.
\]

Conservation Laws

Energy

Functions of the generalized coordinates and speeds that do not change during the motion of a system are integrals of the motion. There are \( 2s - 1 \) such quantities for a system of \( s \) particles, because the choice of starting time is arbitrary and so a parameter \( t + t_0 \) can be eliminated from the \( 2s \) equations \( q_i, \dot{q}_i \), functions of \( t + t_0 \) and \( 2s - 1 \) arbitrary constants expressible as functions of the \( q_i \) and \( \dot{q}_i \), where these functions are integrals of the motion. The quantities that are the integrals of motion are conserved and are also additive.
Momentum

Substituting this into the equation for the energy,

\[ F \frac{d}{dt} \sum_i \mathbf{q}_i \frac{\partial \mathbf{F}_i}{\partial \mathbf{q}_i} = \sum_i \mathbf{F}_i \]

and the quantity

\[ E := \sum_i \mathbf{q}_i \frac{\partial \mathbf{F}_i}{\partial \mathbf{q}_i} - \mathcal{L} \]

called the energy, is conserved. From Euler’s theorem on homogeneous functions,

\[ \sum_i \mathbf{q}_i \frac{\partial \mathbf{F}_i}{\partial \mathbf{q}_i} = \sum_i \frac{\partial}{\partial \mathbf{q}_i} (\mathbf{q}_i \mathbf{q}_i) = 2 \mathcal{T}. \]

Substituting this into the equation for the energy,

\[ E = T(q(q)) + U(q) = \frac{1}{2} \sum_a m_a v_a^2 + U(r_1, r_2, \ldots). \]

Momentum

Now considering the homogeneity of space, the Lagrangian should be unchanged by an infinitesimal displacement \( \epsilon \). This change is

\[ \delta \mathcal{L} = \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{q}_a} \delta \mathbf{q}_a = \epsilon \cdot \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{q}_a} = 0. \]

Since \( \epsilon \) is arbitrary, we must have

\[ 0 = \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{q}_a}, \]

so the vector

\[ \mathbf{P} := \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{q}_a}, \]

called the momentum of the system, is conserved (in a closed system). Substituting in the Lagrangian for a closed system,

\[ \mathbf{P} = \sum_a m_a \mathbf{v}_a. \]

Even in the presence of an external field, components of the momentum corresponding to coordinates that do not appear in the potential energy are conserved (space is homogeneous in these directions). For example, the components of the momentum parallel to the plane separating two half-spaces of different potential energy.

From this derivation, we see that in a closed system,

\[ \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{q}_a} = -\sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{q}_a} = \sum_a \mathbf{F}_a = 0, \]

so with two particles, \( \mathbf{F}_1 + \mathbf{F}_2 = 0 \), which is Newton’s third law.

In generalized coordinates, the \( \mathbf{p}_i = \frac{\partial \mathcal{L}}{\partial \mathbf{\dot{q}}_i} \) are the generalized momenta, and the \( \mathbf{F}_i = \frac{\partial \mathcal{L}}{\partial \mathbf{\dot{q}}_i} \) are the generalized forces. Lagrange’s equations can then be written as \( \mathbf{p}_i = \mathbf{F}_i. \)

Center of mass

Applying a Galilean transformation between inertial frames, we see that the momentum transforms like

\[ \mathbf{P} = \sum_a m_a \mathbf{v}_a + \mathbf{V} \sum_a m_a = \mathbf{P}' + \mu \mathbf{V}, \]

where \( \mu = \sum m_a. \)

One can always choose a frame such that \( \mathbf{P}' = 0 \), giving that \( \mathbf{V} = \mathbf{P}/\sum m_a = \sum m_a \mathbf{v}_a/\sum m_a \), which can be written as the total time derivative of

\[ \mathbf{R} = \sum_m m_a \mathbf{r}_a, \]

which is called the center of mass of the system.

By another Galilean transformation, energies are related by

\[ E = E' + \mathbf{V} \cdot \mathbf{P}' + \frac{1}{2} \mu \mathbf{V}^2. \]

My calculation also shows that actions are related by

\[ S = S' + \mathbf{V} \cdot \mathbf{R}' + \frac{1}{2} \mu \mathbf{V}^2 t, \]

as the transformation in the Lagrangian is apparent from the transformation of the energy.

Angular momentum

Another conserved quantity is due to the isotropy of space. Considering an infinitesimal rotation \( \delta \mathbf{\phi} \), we see that a radius vector is displaced by \( \delta \mathbf{r} = \delta \mathbf{\phi} \times \mathbf{r} \), and velocities also change by \( \delta \mathbf{v} = \delta \mathbf{\phi} \times \mathbf{v} \). Given the condition that the Lagrangian is unchanged by the rotation, replacing \( \partial \mathcal{L}/\partial \mathbf{v}_a \) with \( \mathbf{p}_a \), \( \partial \mathcal{L}/\partial \mathbf{\dot{q}}_a \) with \( \mathbf{p}_a \), and applying the rotation, we get

\[ \delta \mathcal{L} = \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{q}_a} \delta \mathbf{q}_a + \frac{\partial \mathcal{L}}{\partial \mathbf{\dot{q}}_a} \delta \mathbf{\dot{q}}_a + \frac{\partial \mathcal{L}}{\partial \mathbf{v}_a} \delta \mathbf{v}_a = 0, \]

but \( \delta \mathbf{\phi} \) is arbitrary, so the vector

\[ \mathbf{M} := \sum_a \mathbf{r}_a \times \mathbf{p}_a \]

called the angular momentum or moment of momentum is conserved in the motion of a closed system.

Applying a Galilean transformation of the position,

\[ \mathbf{M} = \mathbf{M}' + \mathbf{a} \times \mathbf{P}, \]

where \( \mathbf{a} \) is the displacement from the unprimed to the primed frame. When the primed frame is moving with \( \mathbf{a} = \mathbf{V} \), a Galilean transformation of the primed velocity and use of the center of mass \( \mathbf{R} \) gives that the angular momentum transforms like

\[ \mathbf{M} = \mathbf{M}' + \mu \mathbf{R} \times \mathbf{V} = \mathbf{M}' + \mathbf{R} \times \mathbf{P}, \]

analogous to how the momentum and energy transform (having a contribution from the “intrinsic” or “internal” quantity in the frame in which it is at rest and one from the motion of the system as a whole).

As with the momentum, the angular momentum has conserved components when the system is subject to an external field along axes of symmetry of the field (space is isotropic with respect to rotations about these axes). For example, a point source conserves components of angular momentum on all axes passing through the point.
More generally, the component of the angular momentum in any axis (say the $z$-axis) can be found by differentiating the Lagrangian:

$$M_z = \sum a \frac{\partial L}{\partial \dot{r}_z},$$

where the coordinate $\phi$ is the angle of rotation about the $z$-axis. This can be proven directly by expressing $M_z$ and $L$ in cylindrical coordinates. Substitution of this Lagrangian into the above gives the $M_z$ found in cylindrical coordinates.

There are no other additive integrals of motion, so every closed system has seven of these integrals: energy and the three components each of momentum and angular momentum.

### Mechanical similarity

As before, multiplication of the Lagrangian by a constant has no effect on the equations of motion, and this includes when the potential energy is a homogeneous function of the coordinates: $U(a_1r_1, a_2r_2, \ldots) = a^nU(r_1, r_2, \ldots)$, where $a$ is any constant and $k$ is the degree of homogeneity of the function.

If the coordinates are transformed by a factor $a$ and the time by a factor $\beta$, velocities are changed by a factor $a/\beta$ and so the kinetic energy by $a^2/\beta^2$ and the potential energy by $a^k$. For the motion to be unaltered, the condition that this gives a scalar multiplication is that $a^2/\beta^2 = a^k$, so $\beta = a^1/k$. Thus the Lagrangian permits different equations of motion, for a potential energy that is a homogeneous function, following geometrically similar paths given in the time ratio

$$\frac{t'}{t} = \left(\frac{v'}{v}\right)^{1-\frac{1}{k}}.$$

Similarly, other physical quantities must be in ratios that are powers of the length ratio:

$$\frac{v'}{v} = \left(\frac{v'}{v}\right)^{\frac{1}{k}}, \quad \frac{E'}{E} = \left(\frac{v'}{v}\right)^k, \quad \text{and} \quad \frac{M'}{M} = \left(\frac{v'}{v}\right)^{1+\frac{1}{k}}.$$

Given this similarity information, we can make deductions about mechanical systems.

In small oscillations, the potential energy is a quadratic function of the coordinates with $k = 2$, so the period of oscillation is independent of the amplitude of oscillation.

In a uniform field, the potential energy is a linear function of the coordinates with $k = 1$, so $t'/t = (v'/v)^1$, which gives the time of fall under gravity goes by the square root of the initial amplitude.

In Newtonian attraction, the inverse square potential energy is again a homogeneous function with $k = -1$, giving that $t'/t = (v'/v)^{-1/2}$, which when considered with respect to orbits is Kepler’s third law.

When the potential energy is a homogeneous function of the coordinates for a motion that takes place in a finite region of space, the virial theorem relates the time average kinetic and potential energies. Since the kinetic energy is a quadratic function of the velocities, Euler’s theorem on homogeneous functions gives that $\sum v_a \cdot \partial T/\partial v_a = 2T$, and since $\partial T/\partial v_a = p_a$,

$$2T = \sum a p_a \cdot v_a = \frac{d}{dt} \left( \sum a p_a \cdot r_a \right) - \sum a r_a \cdot \dot{p}_a.$$

The mean value of the time derivative $f$ of a bounded function $F$ has mean value zero, since

$$f = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{df}{dt} dt = \lim_{T \to \infty} \frac{F(T) - F(0)}{T} = 0.$$

So because the motion is within a finite region of space, $\sum a p_a \cdot r_a$ is bounded and the mean value of the first term of $2T$ above is zero. Using Newton’s equations to replace $p_a$ by $-\partial U/\partial r_a$,

$$2T = \sum a r_a \frac{\partial U}{\partial r_a},$$

where the right hand side is called the virial of the system. If the potential energy is a homogeneous function of degree $k$ in the $r_a$ then another application of Euler’s theorem gives that

$$2T = kU,$$

which is the result of the virial theorem. In terms of the energy $\bar{E} = E$,

$$T = \frac{KE}{k+2} \quad \text{and} \quad \bar{U} = \frac{2E}{k+2}.$$

So for small oscillations where $k = 2$, $T = \bar{U}$, and for Newtonian interaction where $k = -1$, $2T = -\bar{U}$ and $E = -T$, because the motion can only occur within a finite region of space if the total energy is negative.

### Integration of the Equations of Motion

#### Motion in one dimension

Without even integrating to obtain the equations of motion, information about the system can be found from the energy. For example, the Lagrangian for a system with one degree of freedom generally takes the form

$$L = \frac{1}{2} (a q)^2 - U(q).$$

For example, if $q$ is a Cartesian coordinate $x$, then we can look at the differential equation given by the energy,

$$E = \frac{1}{2} m x^2 + U(x),$$

and separate variables since $\bar{x} = dx/dt$ to get

$$t = \sqrt{\frac{1}{2m} \int \frac{dx}{\sqrt{E - U(x)}}} + C.$$

The kinetic energy is essentially positive, so motion can only take place in regions of space where $U(x) < E$. The points where $U(x) = E$ are the turning points of the system, and if the region of motion is bounded by two such points, the motion is finite, where if there is only one point the motion is infinite. One-dimensional motion within a potential well given by turning points $x_1$ and $x_2$ is finite, so the period $T$ of the oscillation is twice that from $x_1$ to $x_2$:

$$T(E) = \sqrt{\frac{2m}{E}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}.$$

Special function time: by this method, it is easy to show that the period of oscillation of a simple pendulum is

$$T = 4 \sqrt{\frac{L}{g}} K\left\{\sin\left(\frac{1}{2}\phi_0\right)\right\},$$

where $\phi_0$ is the maximum angle from the vertical and

$$K(k) = \int_0^\frac{\pi}{2} \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}}$$

is the complete elliptic integral of the first kind, so the period is approximately

$$T = 2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{1}{16} \phi_0^2 + \cdots \right).$$

Also, the period of oscillation of a particle in a potential well $U = A|x|^n$ is

$$T = \frac{2}{n} \sqrt{\frac{2\pi}{E}} \left(\frac{E}{A}\right)^{1/n} \Gamma(1/n) \Gamma\left(1/2 + 1/n\right),$$

since integral substitution gives a beta function (Euler integral of the first kind)

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$
which can be expressed in terms of gamma functions
\[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt. \]

Note that the relation between \( T \) and \( E \) obeys the law of mechanical similarity from before, with the energy ratio in the power \(-1/2 + 1/n\) to the time ratio.

**Determination of the potential energy from the period of oscillation**

We now solve the reverse problem from that above. Regarding the function which can be expressed in terms of gamma functions, the two-body problem potential, which we set to be zero, giving
\[ D \text{etermination of the potential energy from the period to the time ratio.} \]

Thus the two-body problem is equivalent to that of the motion of a single particle with a single constraint, since \( d\phi/dt \) \( \alpha \text{ and } \beta \) are cyclic, and hence the generalized momentum \( p_\theta = \alpha \text{ and } p_\phi \) of a cyclic coordinate is an integral of the motion. This requirement simplifies the task of integrating the coordinates of motion.

The generalized momentum \( p_\theta = m^2 \phi^2 \) is conserved. Geometrically, \( dE/dt = 1/2 \alpha \cdot \beta \) is the area of the radial sector along a path element, so \( M = 2m \), where \( \beta \) is the sectoral velocity. Thus conservation of momentum implies conservation of the sectoral: equal areas are swept out in equal times (Kepler’s second law). \( M \) in this case is also called the area integral of the motion.

Anyway, the equations of motion are once again found by separating variables in the energy. Expressing \( \dot{\phi} \) in terms of \( M \),
\[ E = \frac{1}{2} m^2 (r^2 + \dot{r}^2) + U(r) = \frac{1}{2} m^2 + \frac{M^2}{2m^2} + U(r), \]
hence
\[ \frac{dr}{dt} = \sqrt{\frac{2}{m} [E - U(r)] - \frac{M^2}{m^2 r^2} + C.} \]

The equation for the angular momentum gives that \( d\phi = M \, dt/mr^2 \), yielding the equation of the path
\[ \phi = \int \frac{M \, dr/r^2}{\sqrt{2m [E - U(r)] - M^2/r^2}} + C. \]

The conservation of \( M \) gives that \( \phi \) varies monotonically with time, since \( \dot{\phi} \) cannot change sign.

The expression for the energy shows that the radial motion is like one-dimensional motion with an effective potential energy \( U_r(r) = U(r) + M^2/2mr^2 \), where the second term is the centrifugal energy. The limits of the radial motion are given by \( U_r(r) = E \). At the radii that satisfy this equation (turning points), \( \dot{r} = 0 \) (though the particle may still have nonzero \( \dot{\phi} \)).

When there is only the constraint that \( r \geq r_{\text{min}} \), the motion is infinite (coming and going). If the range of motion has two limits, the motion lies in the annulus \( r_{\text{min}} \leq r \leq r_{\text{max}} \), being finite. From the equation of the path, the radius vector between the limits turns by
\[ \Delta r = 2 \int_{r_{\text{min}}}^{r_{\text{min}}} \frac{M \, dr/r^2}{\sqrt{2m [E - U(r)] - M^2/r^2}} + C. \]

The motion is generally only closed when the angle is a rational fraction of \( 2\pi \), drawing some sort of flower shape. Otherwise, all points within the annulus will be hit after infinite time. Only for potential energies varying like \( 1/r \) or \( r^2 \) (a space oscillator) do all finite motions occur in closed paths.

At turning points, the sign of the integrands of giving the path and the radius-time relation changes. Thus the path is symmetric about the radius vector to the turning point. Infinite paths can be found by reflecting one half and finite paths by repeating a single segment of the path at all of the turning points.
The centrifugal energy, present when \( M \neq 0 \), becomes infinite by \( 1/r^2 \) as \( r \to 0 \). The particle can only ever reach the center if the potential energy approaches \( -\infty \) sufficiently rapidly:
\[
\frac{1}{2} m \dot{\theta}^2 = E - U(r) - \frac{M^2}{2 m r^2} > 0, \quad \text{so}
\]
\[
r^2 U(r) + \frac{M^2}{2 m} < E r^2,
\]
\[
[r^2 U(r)]_{r \to 0} < \frac{M^2}{2 m}.
\]
Thus \( U(r) \) must tend to \( -\infty \) as \( -a/r^2 \), where \( a > M^2/2m \), or proportional to \( -1/r^2 \), where \( n > 2 \).

**Kepler’s problem**

An important kind of central field is that with a potential energy like \( 1/r, \) as in Newtonian gravity and Coulomb electrostatic interaction.

First consider an attractive field with \( U = -a/r \), where \( a > 0 \). Then the effective potential energy from eliminating \( \theta \) from the Lagrangian, since \( \theta \) is a cyclic coordinate, is
\[
U_e = -\frac{a}{r} + \frac{M^2}{2 m r^2},
\]
which comes from infinity at \( r = 0 \), has a negative minimum, and has a negative tail approaching the \( r \)-axis, looking slightly like a nose. Its minimum value of
\[
U_{e, \text{min}} = -\frac{ma^2}{2M^2}
\]
occurs at \( r = M^2/ma \). The motion is finite for \( E < 0 \) and infinite for \( E > 0 \).

Substituting the potential \( U = -a/r \) into the equations obtained before for a central field and integrating, we get
\[
\phi = \arccos \left[ \frac{(M/r) - (ma/M)}{\sqrt{2mE + \frac{ma^2}{M^2}}} \right] + C.
\]
If we choose the origin so that the constant is zero and let
\[
p = \frac{M^2}{ma} \quad \text{and} \quad e = \sqrt{1 + \frac{2EM^2}{ma^2}},
\]
then the equation for the path can be written as
\[
\frac{p}{r} = 1 + e \cos \phi,
\]
where \( 2p \) is called the *latus rectum* of the orbit and \( e \) the *eccentricity*. With our origin, the point at \( \phi = 0 \) is the point on the path closest to the origin (the *perihe\( \text{l} \)ion*).

When \( E < 0, e < 1 \), so the orbit is finite and an ellipse. From geometry, the major and minor semi-axes of the ellipse are
\[
a = \frac{p}{1-e^2} = \frac{a}{2|E|} \quad \text{and} \quad b = \frac{p}{\sqrt{1-e^2}} = \frac{M}{\sqrt{2m|E|}}.
\]
The major axis of the ellipse depends only upon the energy of the particle, and not upon its angular momentum. When the energy is its least value \( U_{e, \text{min}} = 0 \) and the ellipse becomes a circle. The least and greatest distances from the focus of the ellipse (center of the field) are
\[
r_{\text{min}} = \frac{p}{1+e} = a(1-e) \quad \text{and} \quad r_{\text{max}} = \frac{p}{1-e} = a(1+e),
\]
which could also be found by solving \( U_e(r) = E \) as before.

Integrating the area integral from before, we get that \( 2mf = TM \). The area of the ellipse is \( f = \pi ab \), so we get
\[
T = 2\pi a^{3/2} \frac{m}{\sqrt{\alpha}} = \pi a \frac{m}{2|E|},
\]
which depends only upon the energy of the particle.

For \( E > 0 \), the motion is infinite. If \( E > 0, e > 1 \), so the path is a hyperbola with the origin at its internal focus. The distance of the perihe\( \text{l} \)ion from the focus is
\[
r_{\text{min}} = \frac{p}{e+1} = a(e-1), \quad \text{where}
\]
\[
a = \frac{p}{e+1} = \frac{a}{2E} \]
is the “semi-axis” of the hyperbola.

When \( E = 0, e = 1 \), so the particle moves in a parabola with perihe\( \text{l} \)ion distance \( r_{\text{min}} = p/2 \), which occurs when the particle starts at rest from infinity.

For elliptical orbits, the equations of motion can be parameterized. As previously,
\[
t = \sqrt{\frac{m}{2|E|}} \int_{-\infty}^{r} \frac{d\theta}{r} = \sqrt{\frac{m}{2a^2|E|}} \int_{-\infty}^{r} \frac{d\theta}{\sqrt{a^2 - (r-a)^2}}
\]
\[
= \sqrt{\frac{ma}{a}} \left[ (1-e \cos \xi) \right] + C \]
\[
= \sqrt{\frac{ma^3}{a}} (\xi - e \sin \xi) + C.
\]
If time is measured so that the constant is zero,
\[
r = a(1-e \cos \xi) \quad \text{and} \quad t = \sqrt{\frac{ma^3}{a}} (\xi - e \sin \xi),
\]
where the particle is at the perihe\( \text{l} \)ion at \( t = 0 \). Similar calculations for the hyperbolic orbits give that
\[
r_{\text{min}} = \frac{p}{e-1} = a(e+1),
\]
\[
r = a(e \cosh \xi - 1), \quad \text{and} \quad t = \sqrt{\frac{ma^3}{a}} (e \sinh \xi - \xi),
\]
As a special treat, there also happens to be another integral of the motion that occurs only in this potential (with either sign of \( a \):
\[
\mathbf{v} \times \mathbf{M} + \dot{r} = \frac{r}{r^2},
\]
which is constant because its total time derivative is \( \mathbf{v} \times \mathbf{M} + \dot{r} = (\mathbf{v} \times \mathbf{r} - \mathbf{a} (\mathbf{r} \times \mathbf{v}) \times \mathbf{r}) / r^3 \), or since \( \mathbf{M} = m \mathbf{v} \times \mathbf{v} = m (\mathbf{r} \times \mathbf{v}) - m \mathbf{v} (\mathbf{r} \times \mathbf{v}) \) and \( (\mathbf{v} \times \mathbf{r}) \times \mathbf{v} = 0 \), the equation of motion, the time derivative vanishes. This vector points along the major axis from the focus to the perihe\( \text{l} \)ion, with magnitude \( ae \). It is another one-valued function of the state, like \( E \) and \( M \), which exists due to the *degeneracy* of the motion.

**Collisions Between Particles**

**Disintegration of particles**

When a particle spontaneously disintegrates into two parts (no external forces), conservation of energy gives that
\[
E_i = E_{i1} + \frac{p_{i1}^2}{2m_1} + E_{i2} + \frac{p_{i2}^2}{2m_2},
\]
in the center of mass system (\( C \)), where \( E_{i1} \) and \( E_{i2} \) are the internal energies of the particles, leaving the disintegration energy
\[
\epsilon = E_i - E_{i1} - E_{i2}
\]
\[
= \frac{p_i^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{p_i^2}{2m},
\]
where \( m \) is the reduced mass of the two particles. When the primary particle moves with velocity \( \mathbf{V} \) in the laboratory system (\( L \)), and one of the resulting particles has velocity \( \mathbf{v} \) in the \( L \) system and \( v_0 \) in the \( C \) system, \( \mathbf{v} - \mathbf{V} = v_0 \), so \( v^2 + \mathbf{V}^2 - 2v_0 \mathbf{V} \cos \theta = v_0^2 \), where \( \theta \) is the angle of the disintegrated particle from \( \mathbf{V} \) in \( L \).
Considering the geometry of these vectors in a circle of radius $r_0$, we see that when $V < v_0$, $\theta$ can be any angle, but when $V > v_0$, it cannot exceed the maximum sin $\theta_{\text{max}} = v_0/V$. The geometry further gives $\theta$ in terms of the angle $\theta_0$ of the disintegrated particle in the $C_\text{e}$ system: $\tan \theta = v_0 \sin \theta_0 / (v_0 \cos \theta_0 + V)$. Solving for $\cos \theta_0$, we get

$$\cos \theta_0 = -\frac{V}{v_0} \sin^2 \theta \pm \sqrt{\frac{1 - V^2/v_0^2}{v_0^2}} \sin^2 \theta.$$  

Now considering many particles in random orientations disintegrating into two (in $C$), we see that every resulting particle of the same kind has the same energy, and that the directions of their motion are isotropically distributed. Thus, the fraction of particles entering a solid angle element $d\Omega = d\theta_0/2\pi$. This distribution with respect to $\theta_0$ is obtained from putting $d\Omega = 2\pi \sin \theta_0 d\theta_0$, giving the fraction

$$\frac{1}{2} \sin \theta_0 d\theta_0.$$  

The distributions of direction and energy in the $L$ system are obtained by transformation. Squaring $v = v_0 + V$, we get $v^2 = v_0^2 + V^2 + 2v_0V \cos \theta_0$, so $d(\cos \theta_0) = d(v^2)/2v_0V$. Substituting the angle distribution for $d(\cos \theta_0)$, the distribution in the kinetic energy $T = 1/2m v^2$, for $m$ being either $m_1$ or $m_2$, is

$$\frac{dT}{2m v_0 V}.$$  

As this shows, the kinetic energies are distributed uniformly over $T_{\text{min}} = 1/2m(v_0 - V)^2$ and $T_{\text{max}} = 1/2m(v_0 + V)^2$.

For several disintegration pieces, the maximum energy any one of them can have can be shown to be

$$T_{1,\text{max}} = \frac{(M - m_1)e}{M}.$$  

### Elastic collisions

As we know from before, the initial velocities (in $C$) of the two particles in an elastic collision are $v_{01} = m_2 v/(m_1 + m_2)$ and $v_{02} = -m_1 v/(m_1 + m_2)$, where $v = v_1 - v_2$. If $n_0$ is a unit vector in the direction of $m_1$ after the collision, then the post-collision velocities are

$$\begin{align*}
\mathbf{v}_{10} &= \frac{m_2 v_0}{m_1 + m_2} n_0 \\
\mathbf{v}_{20} &= -\frac{m_1 v_0}{m_1 + m_2} n_0
\end{align*}$$  

since energy and momentum conservation allow the system only to rotate the velocities. In $L$, these velocities have the velocity $V$ of the center of mass added, giving

$$\begin{align*}
\mathbf{v}_1' &= \frac{m_2 v_0}{m_1 + m_2} + \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \\
\mathbf{v}_2' &= -\frac{m_1 v_0}{m_1 + m_2} + \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}
\end{align*}$$  

The direction of $n_0$ depends on the law of interaction of the particles. The momenta are

$$\begin{align*}
\mathbf{p}_1' &= m_1 v_0 + \frac{m_1}{m_1 + m_2} (p_1 + p_2) \\
\mathbf{p}_2' &= -m_2 v_0 + \frac{m_2}{m_1 + m_2} (p_1 + p_2)
\end{align*}$$  

where $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass. Geometric considerations of a circle of radius $mr$ allow further investigation of the motion. Letting $\chi$ be the angle to $n_0$, we find that the angles $\theta_1$ and $\theta_2$ between the original and final velocities of each particle, as well as the final velocities, are

$$\begin{align*}
\tan \theta_1 &= \frac{m_2 v_1' \sin \chi}{m_1 + m_2 \cos \chi} \\
\theta_1 &= \frac{1}{2} (\pi - \chi) \\
\tan \theta_2 &= \frac{m_1 v_0 + m_2 v_2}{1 + m_1 m_2} \\
\theta_2 &= \frac{1}{2} (\pi - \chi)
\end{align*}$$  

$\theta_1 + \theta_2 > \pi/2$ when $m_1 < m_2$ and $\theta_1 + \theta_2 < \pi/2$ when $m_1 > m_2$.

### Scattering

The elastic collision between two particles is equivalent to the deflection of a single particle of mass $m$ in a central field $U(r)$. We know that motion in a central field is symmetric about the line from the origin to the nearest point in the orbit, at angle $\phi_0$ to each of the asymptotes of an infinite motion. Thus the angle $\chi$ of deflection between the directions of the asymptotes is

$$\chi = |\pi - 2\phi_0|.$$  

We know from before that

$$\phi_0 = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{(M/2^2) \, dr}{\sqrt{2m[E - U(r)] - M^2/2^2}}$$  

and that $r_{\text{min}}$ is a zero of the radicand. In terms of the velocity at infinity, $v_0$, and the impact parameter $\rho$, which is the distance from the origin to the line of travel of the particle if there were no field, we see that $E = 1/2mv_0^2$ and $M = m v_0$, and thus

$$\phi_0 = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{(\rho/2^2) \, dr}{\sqrt{1 - (\rho^2/2^2) - (2U/m v_0^2)}}$$  

so we can express $\chi$ in terms of $\phi_0$.

Consider the scattering of a beam of many identical particles with different impact parameters and thus different angles of scattering $\chi$. Let $dN$ be the number of particles scattered per unit time between $\chi$ and $\chi + d\chi$ and $n$ be the number of particles passing in unit time through a unit area of the beam cross-section, where the beam is assumed to be uniform. Then the ratio

$$d\sigma = \frac{dN}{n}$$  

is called the effective scattering cross-section, being determined entirely by the form of the scattering field and not by the beam.

Assuming that the relation of $\chi$ and $\rho$ is one-to-one, such as in the case of a monotonically decreasing $\chi$ with $\rho$, then only particles with $\rho$ between $\rho(\chi)$ and $d\rho(\chi)$ are scattered between $\chi$ and $\chi + d\chi$. The number of such particles is the product of $n$ with the area of the annulus given by $\rho$ and $\rho + d\rho$, so $dN = 2\pi d\rho n$ and therefore the effective scattering cross-section is

$$d\sigma = 2\pi \rho \, d\rho.$$  

This is also

$$d\sigma = 2\pi \rho(\chi) \left| \frac{d\rho(\chi)}{d\chi} \right| d\chi,$$

and since the solid angle between cones with vertical angles $\chi$ and $\chi + d\chi$ is $d\phi = 2\pi \sin \chi \, d\chi$,

$$d\sigma = \frac{\rho(\chi)}{\sin \chi} \left| \frac{d\rho(\chi)}{d\chi} \right| d\phi.$$  

This gives the effective cross-section as a function of the angle of scattering in $C$. To obtain the result in $L$, we express $\chi$ in terms of $\theta$ as in the preceding section, which gives expressions for the scattering cross-section for the incident beam ($\chi$ in terms of $\theta_0$) and that for the particles initially at rest ($\chi$ in terms of $\theta_2$).

### Rutherford’s formula

For a Coulomb field $U = a/r$, we obtain from an elementary integration that

$$\phi_0 = \arccos \frac{a/m v_0^2 \rho}{\sqrt{1 + (a/m v_0^2 \rho)^2}},$$  

whence $\rho^2 = (a^2/m^2 v_0^2) \tan^2 \phi_0$, or since $\phi_0 = 1/2(\pi - \chi)$, $\rho^2 = (a^2/m^2 v_0^2) \cot^2 \chi/2$. Differentiating the expression for $\phi_0$ with respect
to $\chi$ and substituting it into the final results from the last section, we get

$$d\sigma = \pi \left( \frac{a}{m v_{\infty}^2} \right)^2 \frac{\cos \chi/2 d\chi}{\sin^3 \chi/2}$$

$$= \left( \frac{a}{2m v_{\infty}^2} \right)^2 \frac{d\phi}{\sin^3 \chi/2},$$

which is Rutherford’s formula. The effective cross-section is independent of the sign of $a$, so this result is valid for both attractive and repulsive Coulomb fields. In the case of the particles initially at rest, this result when transformed into the lab system gives

$$d\sigma_2 = 2\pi \left( \frac{a}{m v_{\infty}^2} \right)^2 \sin \theta_2 \frac{d\theta_2}{\cos^3 \theta_2}$$

$$= \left( \frac{a}{m v_{\infty}^2} \right)^2 \frac{d\phi_2}{\cos^3 \theta_2}.$$

If the mass $m_2$ of the scattering particle is much larger than that of the scattered particle, then $\chi \approx \theta_1$ and $m \approx m_1$, so that

$$d\sigma_1 = \left( \frac{a}{4E_1} \right)^2 \frac{d\phi_1}{\sin^4 \theta_1/2},$$

where $E_1 = 1/(2m_1 v_{\infty}^2)$ is the energy of the incident particle.

Let us determine the distribution of the scattered particles with respect to the energy lost in the collision. From before, we know that the velocity acquired by the scattering particle $m_2$ in $C$ is $v'_2 = (2m_1/(m_1 + m_2)) v_{\infty} \sin \chi/2$, and thus the energy acquired by $m_2$ and lost by $m_1$ is $\epsilon = 1/(2m_2) v'_2^2 = (2m_1/m_2) v_{\infty}^2 \sin^2 \chi/2$. Expressing $\sin \chi/2$ in terms of $\epsilon$ and substituting into Rutherford’s formula, we get

$$d\sigma = 2\pi \left( \frac{a^2}{m_2 v_{\infty}^2} \right) \frac{d\epsilon}{\epsilon^2},$$

which gives the cross-section as a function of the energy loss $\epsilon$, which varies from zero to $\epsilon_{\max} = 2m^2 v_{\infty}^2 / m_2$.

**Small-angle scattering**

The calculation of the effective cross-section is simplified if only collisions with a large impact parameter are considered, so $U$ is weak and the angles of deflection are small. Taking the $x$-axis in the direction of the initial momentum of the particle $m_1$ and $p'_1$ to be the momentum of the particle after scattering, we see that $\sin \theta_1 = p'_1 y / p'_1$. Approximating by replacing $\sin \theta_1$ with $\theta_1$ and $p'_1$ with the initial momentum $p_1 = m_1 v_{\infty}$,

$$\theta_1 \approx \frac{p'_1 y}{m_1 v_{\infty}}.$$

Since $p_y = F_y$, $p'_1 y = \int_{-\infty}^{\infty} F_y \, dt$, and $F_y = -\partial U/\partial y = -(\partial U/\partial \rho) \, \partial \rho/\partial y = -(\partial U/\partial \rho) \, y/r$. Approximating the motion with no deflection at all with $y \approx \rho$ at $v_{\infty}$, $F_y \approx -(\partial U/\partial \rho) \, \rho/r$ and $dt = dx / v_{\infty}$, giving that

$$p'_1 y = -\rho v_{\infty} \int_{-\infty}^{\infty} \frac{dU}{v_{\infty}} \, \frac{dx}{r \, dr}.$$

Since for a straight path, $r^2 = x^2 + \rho^2$, $r$ will vary from $\infty$ to $\rho$ and back. Thus the integral over $x$ is twice the integral over $r$ and $dr = r \, dr / \sqrt{r^2 - \rho^2}$, and the angle of scattering is

$$\theta_1 = -\frac{2\rho}{m_1 v_{\infty}^2} \int_{-\infty}^{\infty} \frac{dU}{dr} \, \frac{dr}{\sqrt{r^2 - \rho^2}}.$$

Thus the effective cross-section for scattering (in $L$), with $\theta_1$ instead of $\chi$ and $\sin \theta_1 \approx \theta_1$, is

$$d\sigma = \left| \frac{d\rho}{d\theta_1} \right| \frac{\rho(\theta_1)}{\theta_1} \, d\sigma_1.$$