Non-nilpotent elements in motivic homotopy theory

Equivariant music - Steve Reich's "Clapping Music"

$x = (1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0) \in \mathbb{F}_2^{12}$.
$C_{12} \subset \Sigma_{12}$ acts on $\mathbb{F}_2^{12}$. Let $g = (1\ 2\ 3\ \cdots\ 11\ 12) \in C_{12}$.

There are two players and the score is:

1. $(x, x) \in \mathbb{F}_2^{12} \times \mathbb{F}_2^{12}$
2. $(x, xg) \in \mathbb{F}_2^{12} \times \mathbb{F}_2^{12}$
3. $(x, xg^2) \in \mathbb{F}_2^{12} \times \mathbb{F}_2^{12}$
4. $(x, xg^3) \in \mathbb{F}_2^{12} \times \mathbb{F}_2^{12}$
5. ...
6. $(x, xg^{12}) = (x, x) \in \mathbb{F}_2^{12} \times \mathbb{F}_2^{12}$.

Question: What choices of $x \in \mathbb{F}_2^{12}/C_{12}$ lead to interesting pieces of music?

Claim: Apparently only two choices! Reich chose a particular representative of one of them.

Chromatic homotopy theory at $p = 2$

First, I would like to recall the theme of chromatic homotopy theory, which provides a beautiful framework for making calculations.

Throughout this talk we will work at the prime 2.

1. All classical spectra will be 2-local.
2. All motivic spectra will be completed with respect to the Eilenberg-MacLane spectrum $H\mathbb{F}_2$.

Non-nilpotent self-maps

Suppose X is a finite 2-local spectrum and $f : X \to \Sigma^{-d}X$ is a non-nilpotent self-map. By definition, this means that each composite

$$X \to \Sigma^{-d}X \to \Sigma^{-2d}X \to \ldots \to \Sigma^{-(n-1)d}X \to \Sigma^{-nd}X$$

is nontrivial.

What does this mean for $\pi_*(X)$?

1. There are f-free elements. Let $f^{-1}X$ be the homotopy colimit of
$$X \to \Sigma^{-d}X \to \Sigma^{-2d}X \to \ldots$$
and try to calculate $\pi_*(f^{-1}X)$.
2. There are f-torsion elements. Let X/f be the cofiber of f. Try to calculate $\pi_*(X/f)$. Do this by trying to find a non-nilpotent self-map on X/f and iterating the above procedure.

Example

$2 : S^0 \to S^0$ is a non-nilpotent self-map.

1. The only 2-free elements are in stem 0. Serre's calculation says
$$\pi_*(2^{-1}S^0) = \mathbb{Q}.$$
2. We'd like to calculate $\pi_*(S/2)$.

$$\begin{tikzcd}
\Sigma^8 S/2 \arrow[r, "v_1^4"] & S/2 \arrow[d] \\
S^8 \arrow[u] \arrow[r, "8\sigma"] & S^1 \arrow[d, "2"] \\
S^8 \arrow[u, "2"] & S^1
\end{tikzcd}$$

v_1^4 induces an isomorphism on K-theory, so is non-nilpotent. Mahowald calculates $\pi_*(v_1^{-1}S/2)$.

Periodicity

Theorem (Mitchell, Hopkins, Smith)

For each $n \geq 0$, there exists a finite spectrum X such that

$$K(n)_* X \neq 0, \ K(n-1)_* X = 0.$$

Any such spectrum admits a self map f such that $K(n)_* f$ is an isomorphism and $K(j)_* f$ is nilpotent whenever $j \neq n$. Such an f is called a v_n self-map.

Examples:

1. $n = 0$: $2 : S^0 \to S^0$
2. $n = 1$: $v_1^4 : S/2 \to \Sigma^{-8} S/2$
3. $n = 2$: $v_2^{32} : S/(2, v_1^4) \to \Sigma^{-192} S/(2, v_1^4)$

Theorem says we can always continue this process.

Motivic homotopy theory

The spaces of topology + the schemes of algebraic geometry
$\rightsquigarrow$ motivic homotopy theory.

Have usual topological 1-sphere $S^{1,0} = S^1$
and the algebraic 1-sphere $S^{1,1} = \mathbb{A}^1 - 0$.

Must specify a ground field. For most of the talk it's $\mathbb{C}$.

New question: what is $\pi_{*,*}(S^{0,0})$?
Second grading is called weight.
Realization functor $X \mapsto X(\mathbb{C})$ returns classical story.

Motivic η

Recall, classically

$$\eta : S^3 \subset \mathbb{C}^2 - 0 \to \mathbb{P}^1(\mathbb{C}) = S^2$$

so $\eta \in \pi_1(S^0)$.

Motivically,

$$\eta : S^{3,2} = \mathbb{A}^2 - 0 \to \mathbb{P}^1 = S^{2,1}$$

so $\eta \in \pi_{1,1}(S^{0,0})$.

Remarkably $\eta^n \neq 0$ for all n in the motivic story.
We'll see this by a spectral sequence argument.

Program

1. $\pi_{*,*}(\eta^{-1}S^{0,0})$ over $\mathbb{C}$.
2. A self-map on S/η over $\mathbb{C}$. Then over $\mathbb{R}$.
3. Over $\mathbb{R}$: towards an analog of the theorem of Mitchell, Hopkins and Smith.

The Adams-Novikov spectral sequence

$$H^{s,u}(BP_*BP) \overset{s}{\Longrightarrow} \pi_{u-s}(S^0) \otimes \mathbb{Z}_{(2)}.$$

$BP_*BP = \pi_*(BP \wedge BP)$,
$BP =$ the 2-local Brown-Peterson spectrum.

$H(BP_*BP) =$ cohomology of the Hopf algebroid BP_*BP with coefficents in BP_*, a $\mathbb{Z}_{(2)}$-algebra.

$\pi_*(S^0)$ is filtered.
$F^s\pi_{u-s}(S^0)/F^{s+1}\pi_{u-s}(S^0)$ approximated by $H^{s,u}(BP_*BP)$.

The motivic Adams-Novikov spectral sequence

Non-zero elements of $H^{s,u}(BP_*BP)$ have u even and so we can assign them a weight $w = u/2$.

The motivic Adams-Novikov spectral sequence takes the form

$$H^{s,u}(BP_*BP)[\tau]^w \overset{s}{\Longrightarrow} \pi_{u-s,w}((S^{0,0})^\wedge_2)$$

where $|\tau| = (0, 0, -1)$. Hu, Kriz, Ormsby, Dugger, Isaksen.

Setting $\tau = 1$ and forgetting the weight returns the classical spectral sequence.

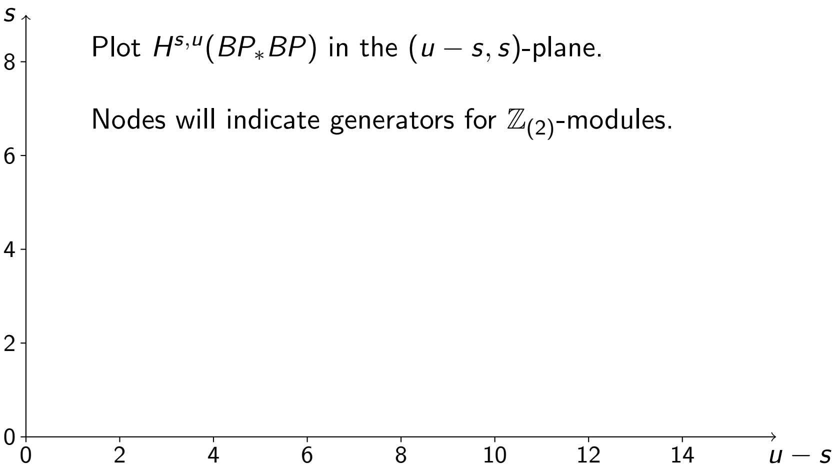
Classical differentials give motivic differentials:

$$d_{2n+1}x = y \rightsquigarrow d_{2n+1}x = \tau^n y.$$

The Adams-Novikov spectral sequence

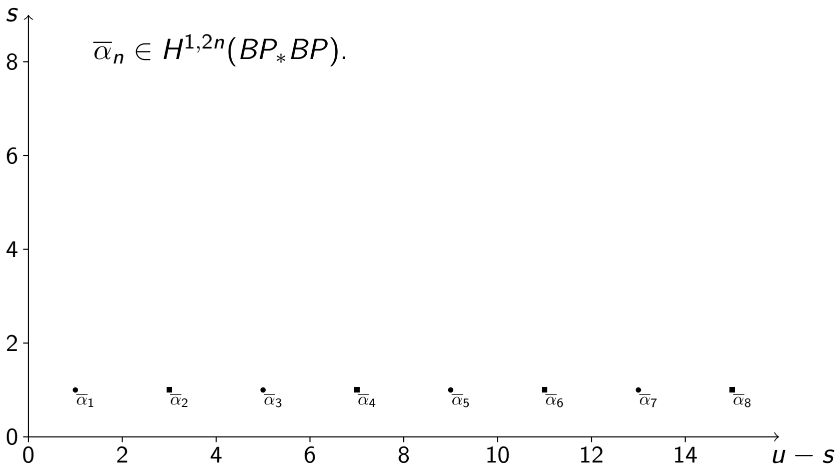
Plot $H^{s,u}(BP_*BP)$ in the $(u-s,s)$-plane.

Nodes will indicate generators for $\mathbb{Z}_{(2)}$-modules.



The Adams-Novikov spectral sequence

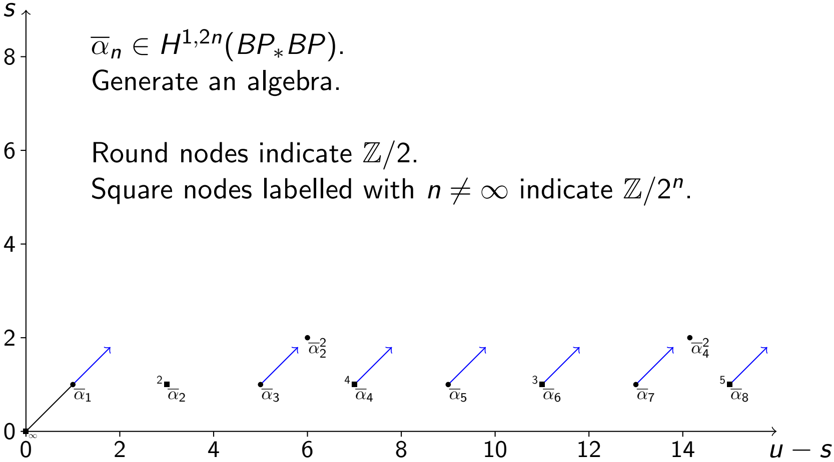
$\overline{\alpha}_n \in H^{1,2n}(BP_*BP)$.



The Adams-Novikov spectral sequence

$\overline{\alpha}_n \in H^{1,2n}(BP_*BP)$.
Generate an algebra.

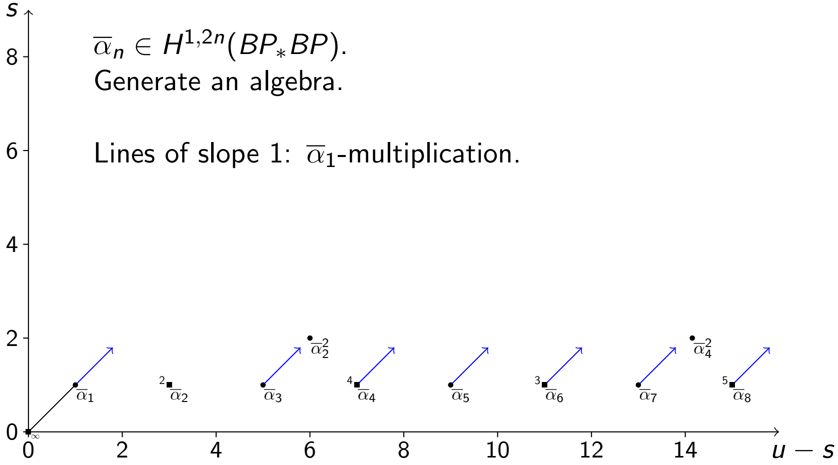
Round nodes indicate $\mathbb{Z}/2$.
Square nodes labelled with $n \neq \infty$ indicate $\mathbb{Z}/2^n$.



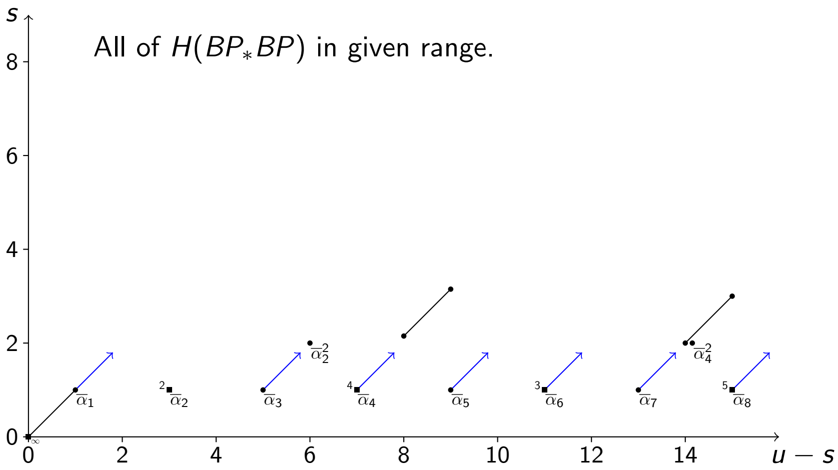
The Adams-Novikov spectral sequence

$\overline{\alpha}_n \in H^{1,2n}(BP_*BP)$.
Generate an algebra.

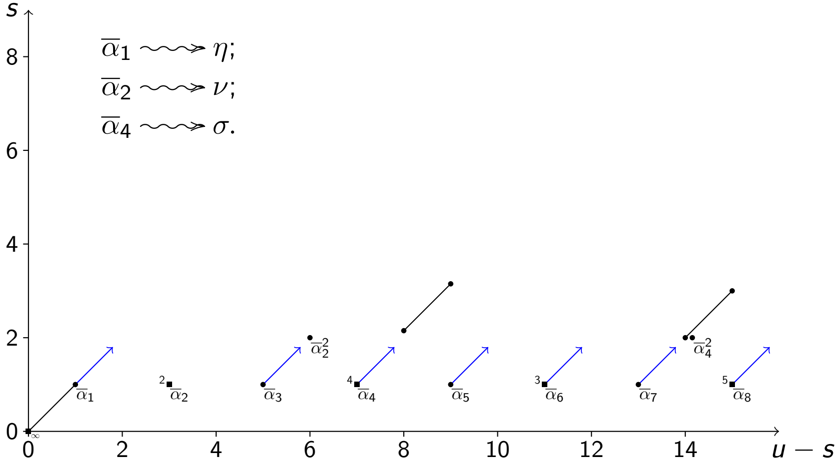
Lines of slope 1: $\overline{\alpha}_1$-multiplication.



The Adams-Novikov spectral sequence



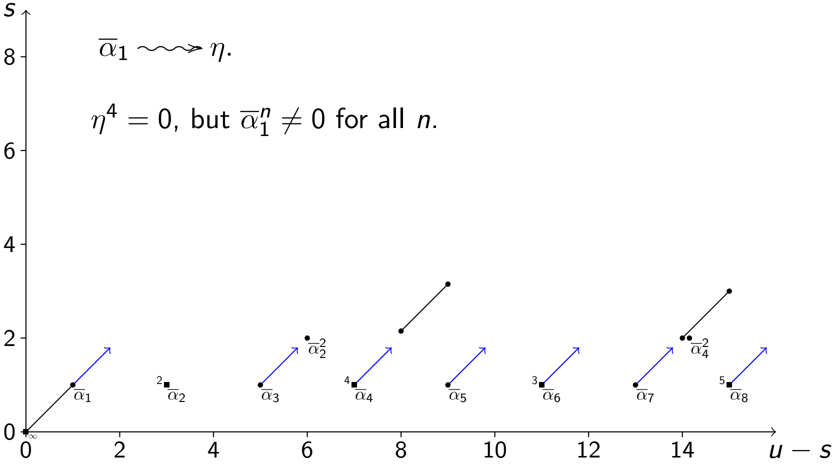
The Adams-Novikov spectral sequence



The Adams-Novikov spectral sequence

$\overline{\alpha}_1 \rightsquigarrow \eta.$

$\eta^4 = 0$, but $\overline{\alpha}_1^n \neq 0$ for all n.

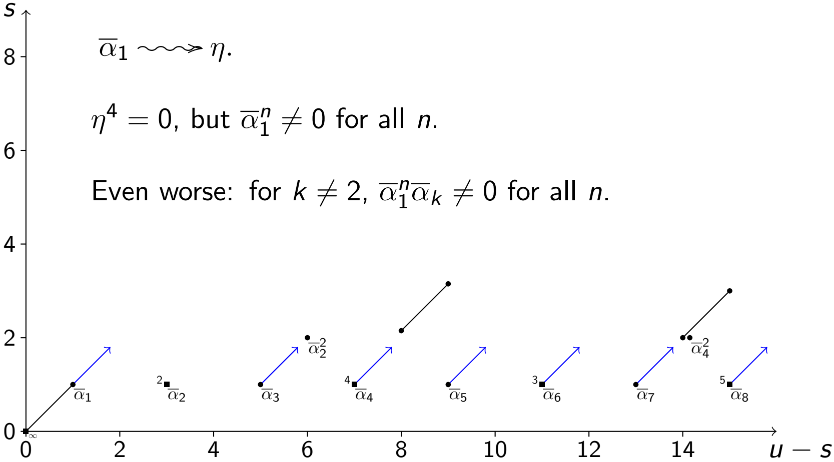


The Adams-Novikov spectral sequence

$\overline{\alpha}_1 \rightsquigarrow \eta.$

$\eta^4 = 0$, but $\overline{\alpha}_1^n \neq 0$ for all n.

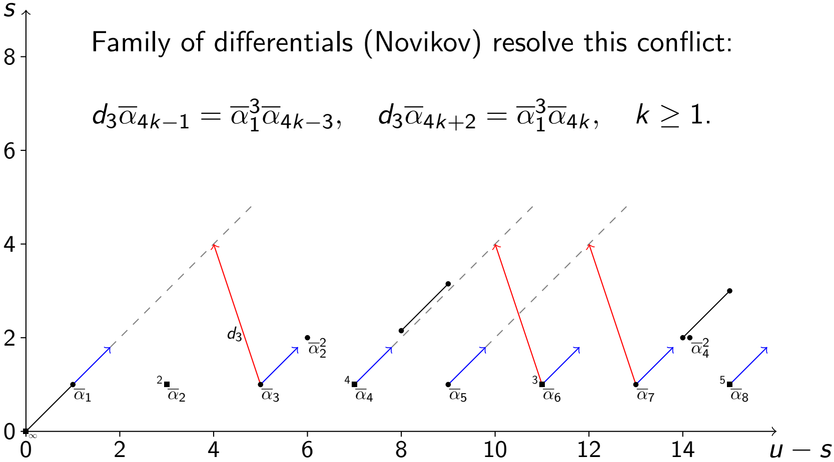
Even worse: for $k \neq 2$, $\overline{\alpha}_1^n \overline{\alpha}_k \neq 0$ for all n.



The Adams-Novikov spectral sequence

Family of differentials (Novikov) resolve this conflict:

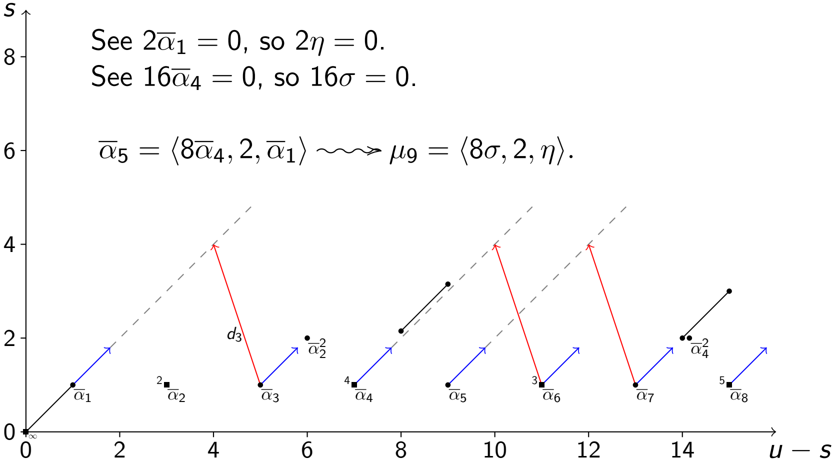
$$d_3\overline{\alpha}_{4k-1} = \overline{\alpha}_1^3\overline{\alpha}_{4k-3}, \quad d_3\overline{\alpha}_{4k+2} = \overline{\alpha}_1^3\overline{\alpha}_{4k}, \quad k \geq 1.$$



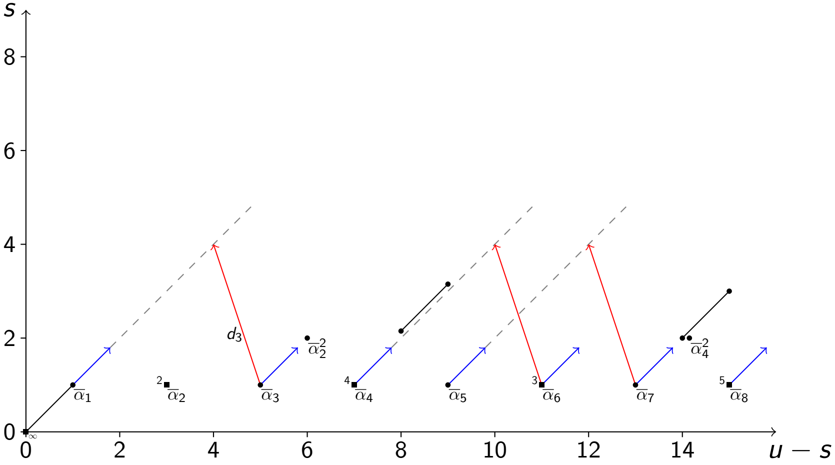
The Adams-Novikov spectral sequence

See $2\overline{\alpha}_1 = 0$, so $2\eta = 0$.
See $16\overline{\alpha}_4 = 0$, so $16\sigma = 0$.

$$\overline{\alpha}_5 = \langle 8\overline{\alpha}_4, 2, \overline{\alpha}_1 \rangle \rightsquigarrow \mu_9 = \langle 8\sigma, 2, \eta \rangle.$$

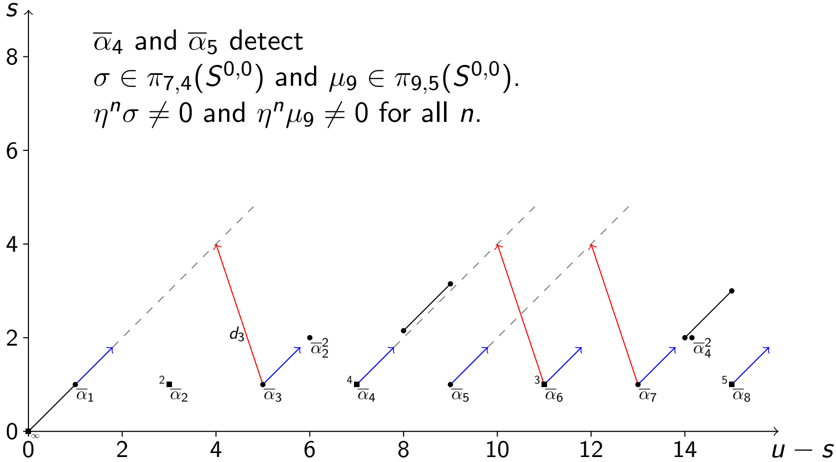


The (motivic) Adams-Novikov spectral sequence



The motivic Adams-Novikov spectral sequence

$\overline{\alpha}_4$ and $\overline{\alpha}_5$ detect
$\sigma \in \pi_{7,4}(S^{0,0})$ and $\mu_9 \in \pi_{9,5}(S^{0,0})$.
$\eta^n\sigma \neq 0$ and $\eta^n\mu_9 \neq 0$ for all n.



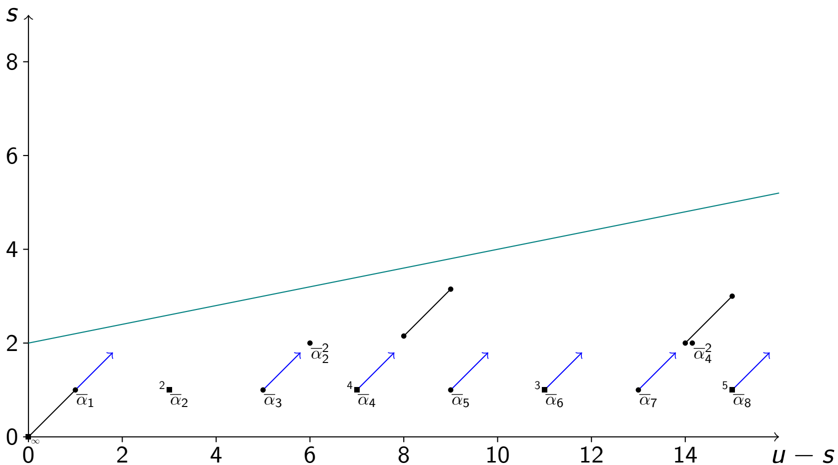
The η-local homotopy of the motivic sphere

Theorem (A., Miller)

$$\pi_{*,*}(\eta^{-1} S^{0,0}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2),$$

where $\eta \in \pi_{1,1}(S^{0,0})$ *and* $\sigma \in \pi_{7,4}(S^{0,0})$ *are motivic Hopf invariant one elements and* $\mu_9 = \langle 8\sigma, 2, \eta \rangle \in \pi_{9,5}(S^{0,0})$ *is detected by* $\overline{\alpha}_5$ *in the motivic Adams-Novikov spectral sequence.*

The Adams-Novikov E_2-page



Key computation: $\overline{\alpha}_1^{-1} H(BP_* BP)$

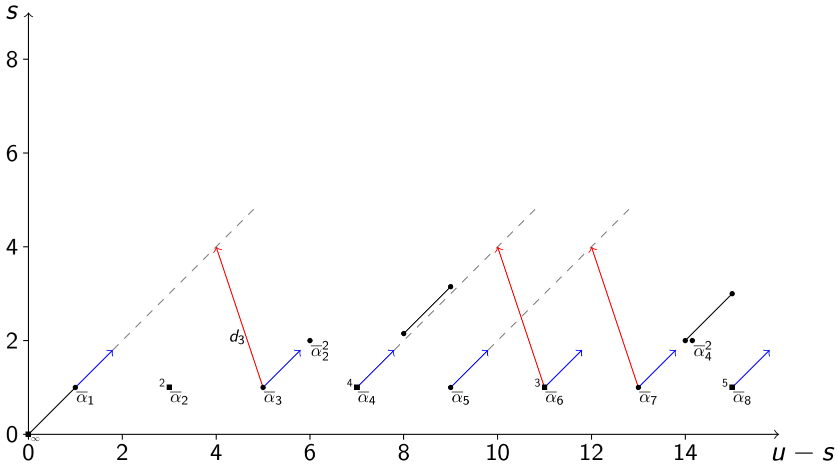
Proposition (A., Miller)

$$H(BP_* BP) \to \overline{\alpha}_1^{-1} H(BP_* BP)$$

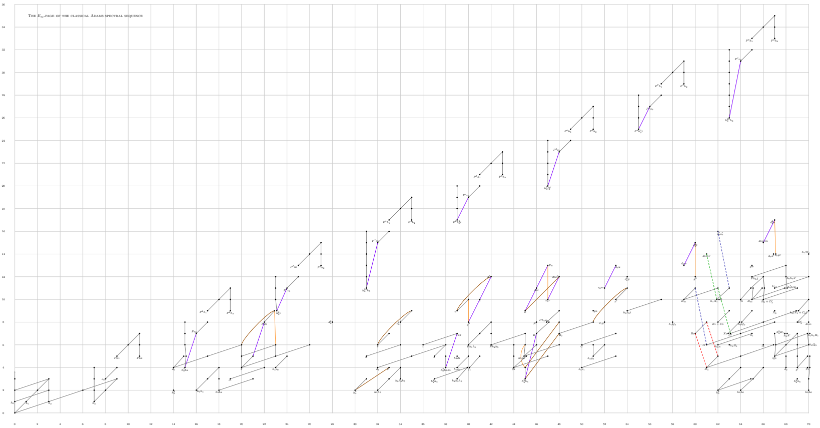
is an isomorphism above a line of slope $1/5$ *in* $(u - s, s)$ *coordinates and*

$$\overline{\alpha}_1^{-1} H(BP_* BP) = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1 \overline{\alpha}_4^2).$$

The (motivic) Adams-Novikov spectral sequence



The chromatic approach to classical homotopy theory



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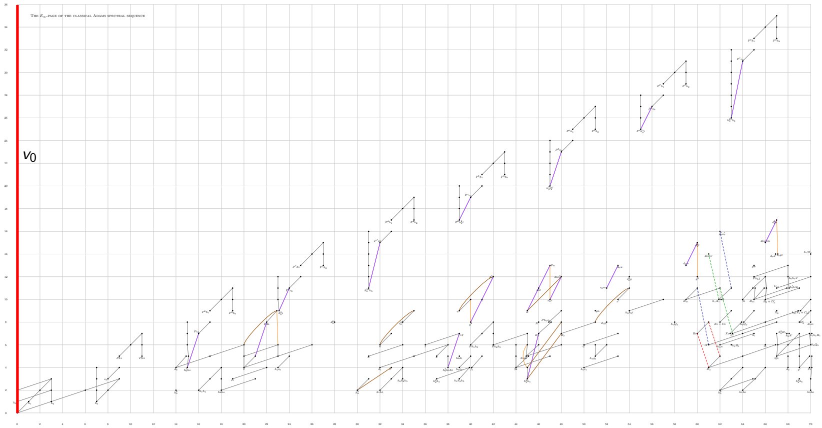
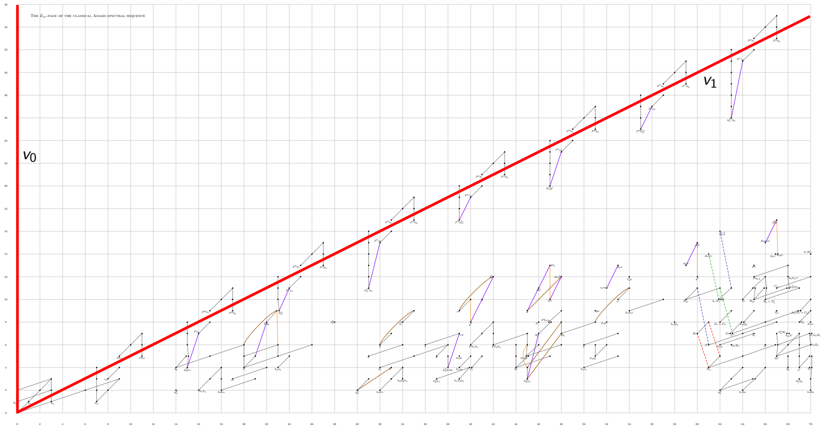


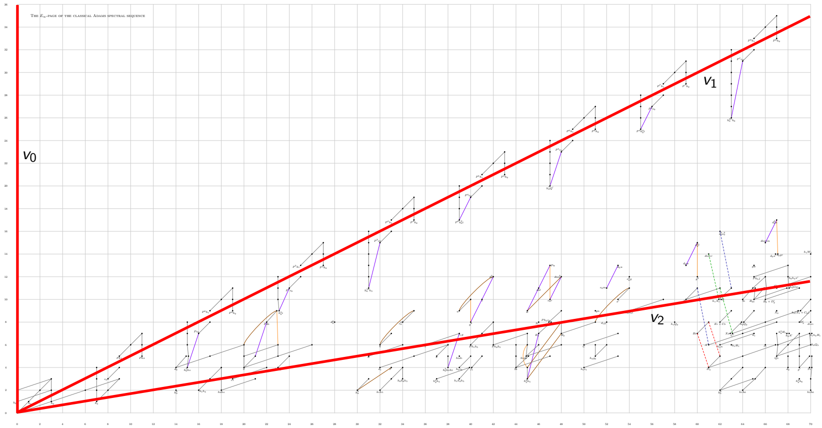
1. $2 : S^0 \to S^0$, $\pi_*(2^{-1}S^0)$;

The chromatic approach to classical homotopy theory



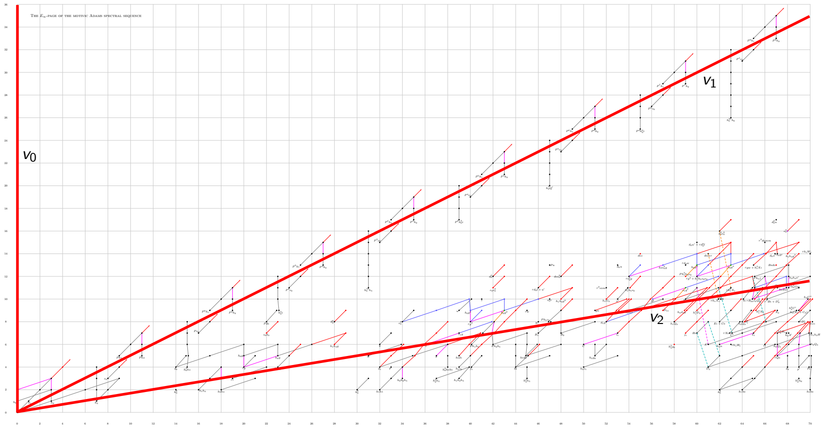
1. $2 : S^0 \to S^0,$ $\qquad \pi_*(2^{-1}S^0);$
2. $v_1^4 : S/2 \to \Sigma^{-8}S/2,$ $\qquad \pi_*(v_1^{-1}S/2);$

The chromatic approach to classical homotopy theory



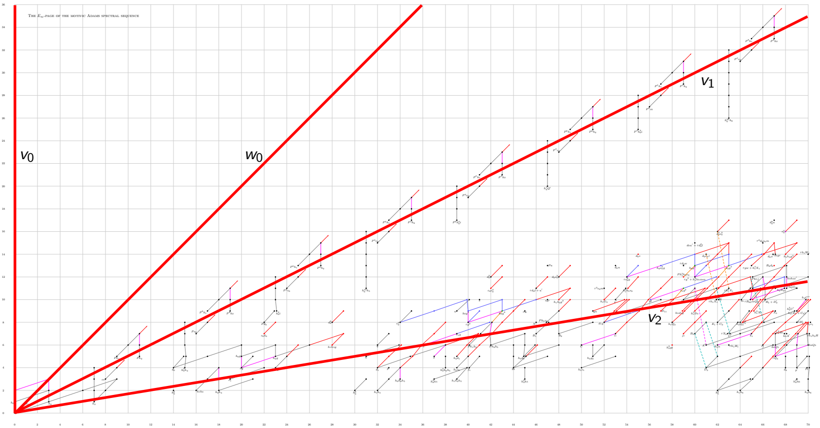
1. $2 : S^0 \to S^0$, $\quad \pi_*(2^{-1}S^0)$;
2. $v_1^4 : S/2 \to \Sigma^{-8} S/2$, $\quad \pi_*(v_1^{-1}S/2)$;
3. $v_2^{32} : S/(2, v_1^4) \to \Sigma^{-192} S/(2, v_1^4)$, $\quad \pi_*(v_2^{-1}S/(2, v_1^4))$.

The chromatic approach to motivic homotopy theory



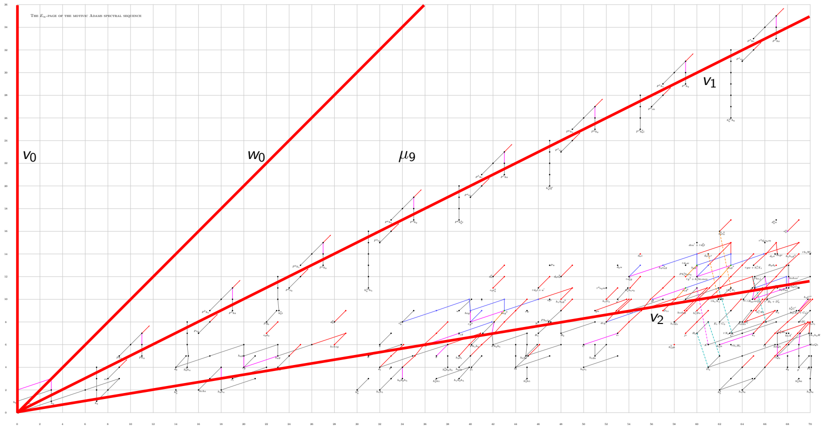
1. $\pi_{*,*}(2^{-1} S^{0,0})$;
2. $\pi_{*,*}(v_1^{-1} S/2)$;
3. $\pi_{*,*}(v_2^{-1} S/(2, v_1^4))$.

But is there more?



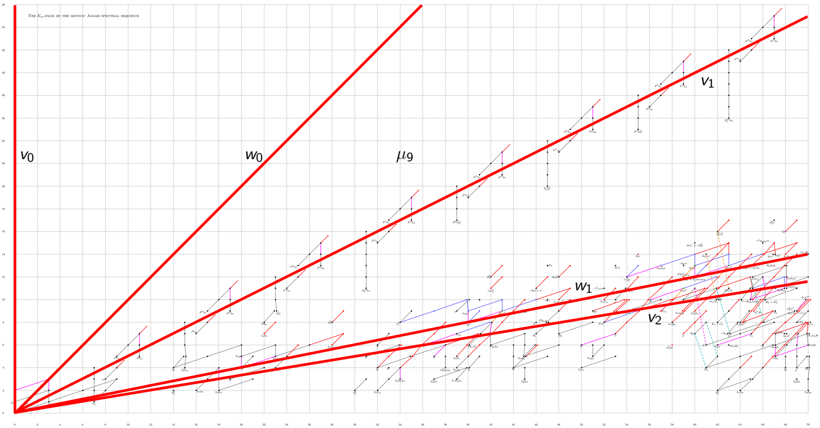
1. $w_0 = \eta : S^{0,0} \to \Sigma^{-1,-1} S^{0,0}$.

But is there more?



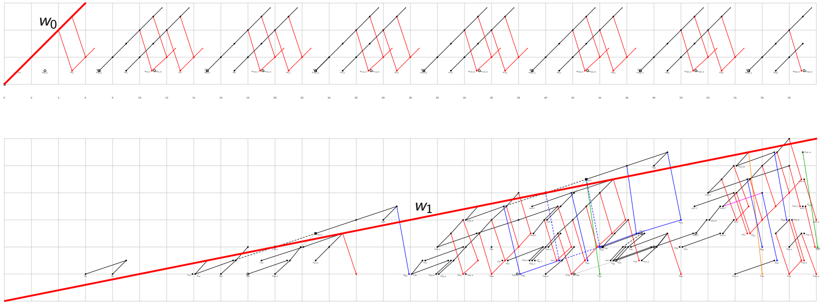
1. $w_0 = \eta : S^{0,0} \to \Sigma^{-1,-1} S^{0,0}$.
2. $\mu_9 \in \pi_{9,5}(S^{0,0})$ is v_1 and w_0-periodic. It fills the gap.

But is there more?



1. $w_0 = \eta : S^{0,0} \to \Sigma^{-1,-1} S^{0,0}$.
2. $\mu_9 \in \pi_{9,5}(S^{0,0})$ is v_1 and w_0-periodic. It fills the gap.
3. $w_1^4 : S/\eta \to \Sigma^{-20,-12} S/\eta$?

But is there more? A new constellation



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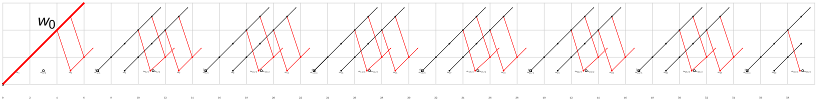
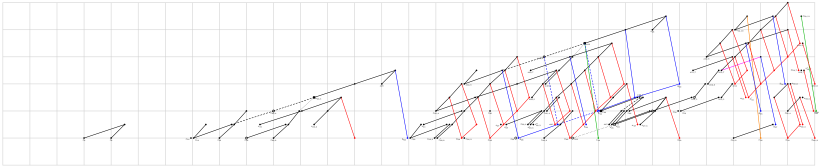
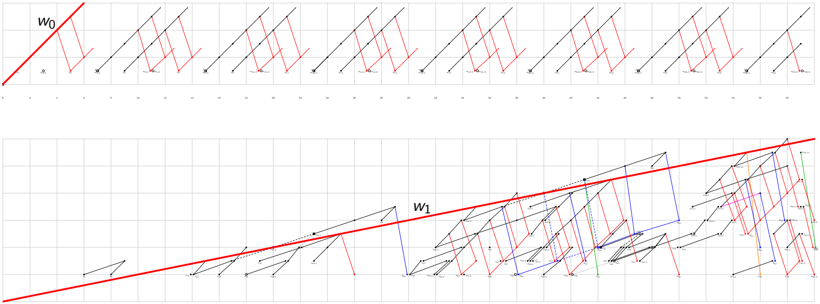




But is there more? A new constellation



A new non-nilpotent self-map

Classically, we have non-nilpotent self-map

$$v_1^4 : \Sigma^8 S/2 \to S/2,$$

Motivically, we have...

Theorem (A.)

There is a non-nilpotent self map

$$w_1^4 : \Sigma^{20,12} S/\eta \to S/\eta.$$

What's the map?

$$
\begin{tikzcd}
\Sigma^8 S/2 & S/2 \arrow[d] \\
S^8 \arrow[u] \arrow[r, "8\sigma"] & S^1 \arrow[d, "2"] \\
S^8 \arrow[u, "2"] & S^1
\end{tikzcd}
$$

What's the map?

$$\begin{tikzcd}
\Sigma^8 S/2 \arrow[r, "v_1^4"] & S/2 \arrow[d] \\
S^8 \arrow[u] \arrow[r, "8\sigma"] & S^1 \arrow[d, "2"] \\
S^8 \arrow[u, "2"] & S^1
\end{tikzcd}$$

What's the map?

$$\begin{tikzcd}
\Sigma^{20,12} S/\eta & & S/\eta \arrow[d] \\
S^{20,12} \arrow[u] \arrow[rr, "\eta^2\eta_4"] & & S^{2,1} \arrow[d, "\eta"] \\
S^{21,13} \arrow[u, "\eta"] & & S^{1,0}
\end{tikzcd}$$

What's the map?

$$
\begin{tikzcd}
\Sigma^{20,12} S/\eta \arrow[r, "w_1^4"] & S/\eta \arrow[d] \\
S^{20,12} \arrow[u] \arrow[r, "\eta^2\eta_4"] & S^{2,1} \arrow[d, "\eta"] \\
S^{21,13} \arrow[u, "\eta"] & S^{1,0}
\end{tikzcd}
$$

My secret: a square of SSs

Classically, one has a diagram:

$$\begin{tikzcd}
H(P;Q) \arrow[r, Rightarrow, "\text{CESS}"] \arrow[d, Rightarrow, "\text{alg-Nov-SS}"'] & H(A) \arrow[d, Rightarrow, "\text{ASS}"] \\
H(BP_*BP) \arrow[r, Rightarrow, "\text{ANSS}"] & \pi_*(S^0)
\end{tikzcd}$$

Motivicifying the square

$$
\begin{tikzcd}
H(P;Q)[\tau] \arrow[r, Rightarrow, "\text{CESS}"] \arrow[d, Rightarrow, "\text{alg-Nov-SS}[\tau]"'] & H(A_{\mathrm{mot}}) \arrow[d, Rightarrow, "\text{MASS}"] \\
H(BP_*BP)[\tau] \arrow[r, Rightarrow, "\text{MANSS}"] & \pi_{*,*}(S^{0,0})
\end{tikzcd}
$$

We localized with respect to an element $h_0 \in H(P;Q)[\tau]$ and completely described the square.

We did not only compute $\pi_{*,*}(\eta^{-1}S^{0,0})$ but also the localized motivic ASS converging to it, confirming a conjecture of Guillou-Isaksen.

A main ingredient in the proof is a formula due to Miller and Wilson in their first paper on Novikov's Ext^1 modulo an invariant prime ideal.

The classical chromatic story and Q

The classical chromatic story is loaded up in Q:

$$q_n \in Q \rightsquigarrow v_n \in BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots].$$

Everything one does with BP_*

e.g. $BP_*/(p, v_1, \ldots, v_{n-1})$, $v_n^{-1}BP_*/(p, v_1, \ldots, v_{n-1})$,
$v_n^{-1}BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty)$

one can do with Q and one has appropriate algebraic Novikov SSs.

A new chromatic story with P?

$h_0 \in H(P; Q)$ corresponds to $\xi_1 \in P$.

Classically, the chromatic story is governed by $p, v_1, v_2, \ldots \in BP_*$ and thus, $q_0, q_1, q_2, \ldots \in Q$.

Motivically, there may be other periodicity operators corresponding to $\xi_1, \xi_2, \xi_3, \ldots \in P$.

I want to call them $\eta_0, \eta_1, \eta_2, \ldots$
Mahowald already has the η_j family so I'll go with $w_0, w_1, w_2, \ldots$

How to make sense of w_n? Over $\mathbb{R}$.

Given a motivic spectrum X defined over $\mathbb{R}$, its complex points define a genuine Σ_2-spectrum, and we can take geometric fixed points $\Phi^{\Sigma_2} X_{\mathbb{C}}$.

1. $\eta : S^{1,1} \to S^{0,0}$ gives rise to $2 : S^0 \to S^0$.
2. $w_1^4 : \Sigma^{20,12} S/\eta \to S/\eta$ can be defined over $\mathbb{R}$. The construction returns the Adams self-map $v_1^4 : \Sigma^8 S/2 \to S/2$.
3. By a w_n self-map, we'll mean a map defined over $\mathbb{R}$ such that the geometric fixed points of the map on complex points is a v_n self-map.

Toward a periodicity theorem

Theorem

For each $n \geq 0$, there exists a finite spectrum X defined over $\mathbb{R}$ such that

$$K(n)_*(\Phi^{\Sigma_2} X_{\mathbb{C}}) \neq 0, \;\; K(n-1)_*(\Phi^{\Sigma_2} X_{\mathbb{C}}) = 0.$$

We can chose such a spectrum so that it admits a w_n self-map.

The end: thank you for listening