MATH 412: TOPICS IN ALGEBRA
FINAL PROJECT

In your final project in Math 412, you will explore and explain a special topic in Galois theory through a 7-10 page paper and a 20-minute presentation.

Topic selection and proposal. Appendix C of Cox contains seventeen additional topics in Galois theory that are appropriate for your project. Here are several more:

» Connections with the fundamental group. There are tight analogies between covering space theory in topology and the Galois correspondence. If you have seen the formner topic in a topology course, formalize this analogy. This leads to Grothendieck's theory of the fundamental group in algebraic geometry. Related topic: Galois theory of étale algebras. (See Milne's course notes on field and Galois theory.)

» Infinite Galois theory. Galois theory naturally extends to infinite algebraic extensions if you take into account the profinite topology on the Galois group. A very important (but still very mysterious) example is $\overline{\mathbb{Q}}/\mathbb{Q}$, the algebraic closure of $\mathbb{Q}$ over $\mathbb{Q}$. An approachable and still very interesting example is $\overline{\mathbb{F}}_p/\mathbb{F}_p$, which has Galois group $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$.

» The transcendence of e and π. This results follow from the the following fact: Given $\alpha_1, \ldots, \alpha_n$ all algebraic and linearly independent over $\mathbb{Q}$, $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$. (Do you see how?) You can find a proof in Stewart's Galois theory book.

» Hilbert's Satz 90. This famous and relatively elementary theorem states an important fact about norms and Galois extensions. It leads to Galois cohomology, Kummer theory, and Artin-Schreier theory, all important next topics in Galois theory which could be exposited through the lens of this theorem.

» Ordered fields, real closed fields, and the Artin-Schreier theorem. Order theory for fields replaces the need for real embeddings in many classical theorems. The Artin-Schreier theorem says that every ordered field has a *real closure* R, a formally real extension in which every polynomial of odd degree with coefficients in R has at least one root in R, and for every element $a \in R$ there exists $b \in R$ such that $a = b^2$ or $a = -b^2$. Such real closed fields have the same first-order properties as $\mathbb{R}$!

» Choose your own adventure. With instructor approval, you may work on a different topic related to Galois theory. *Note for seniors*: The topic must be distinct from your thesis topic.

You must propose first and second choice topics by 26 October. Please send an email to me listing your preferences, along with one or two setences describing what you would cover and one or two resources you might draw from. Topics will be assigned first-come-first-serve to avoid duplication.

Final paper. You will write a 7-10 page paper on your selected topic. You must write the paper using LaTeX in the `amsart` document class using the page layout specified in the template file `http://people.reed.edu/~ormsbyk/412/paper_template.tex`. Resources for learning LaTeX are available on the course website.

You must submit a draft (at least four pages long) at the start of class on 16 November. You will receive comments on your draft by 19 November. After incorporating these comments, you will **submit your final paper at the start of class on 26 November.**

The audience for your paper is a contemporaneous Math 412 student. Your paper should create interest in the topic you are exploring, explain its context, and present some of the pertinent results with proofs. You should assume the material presented in class, but nothing beyond. Please use LaTeX's sectioning, numbering, and environment features to format your paper in typical mathematical style; please use BibTeX for your bibliography.

Final presentation. You will also give a 20-minute presentation on your topic in class on an assigned date between 26 November and 5 December. You will schedule a practice talk with me before your talk and are encouraged to incorporate feedback from that meeting into your talk. The talk may be chalk- or slide-based. (You can do LaTeX slides via the `beamer` package, but there are other options as well.) Given the time constraints, your talk should focus on concepts and theorem statements, but you should also sketch at least one proof. Finally, you will assign one homework problem to the class based on your topic.

You will also act as the audience for your classmates' presentations. In this role, you will provide written feedback to presenters and complete their homework problems.

Learning goals and assessment. As you progress as a mathematician, you will need to independently learn and communicate material, in both written and verbal forms. Your final project will provide a structure in which to practice these skills, including LaTeX document preparation and presentation skills pertinent to your Reed thesis work. By submitting a draft paper and giving a practice presentation, you will have the opportunity to learn from initial mistakes and improve your material.

These papers and presentations are also an opportunity for the class to share its interests with each other and for all of us to learn about additional topics in Galois theory. For some of you, it will be the first of many opportunities to teach mathematics to others. You are expected to participate as an active audience member, to provide feedback on your peers' presentations, and to complete/attempt their homework problems.

Your final paper will be assessed based on the following characteristics (in roughly descending order of importance):

» Mathematical content and precision.
» Mathematical context and narrative.
» Style, including clarity and grammar.
» LaTeX formatting.

Your presentation will be assessed based on the following characteristics (again in roughly descending order of importance):

» Clarity of ideas and information.
» Organization and narrative.
» Clarity of board work or slides.
» Speaking volume and body language.
» Response to audience questions.
» Appropriateness of homework problem.

You will also receive holistic comments on the overall quality of your paper and presentation.