## Solving Polynomials by Radicals

* Assume all fields of char 0. *

**Defn** Let $f \in F[x]$ be nonconstant with splitting field $L/F$.

(a) A root $\alpha \in L$ of $f$ is expressible by radicals over $F$ if $\alpha$ lies in some radical extension of $F$.

(b) The polynomial $f$ is solvable by radicals over $F$ if $L/F$ is a solvable extension.

**Prop** Let $f \in F[x]$ be irreducible. Then $f$ is solvable by radicals over $F$ iff $f$ has a root expressible by radicals over $F$.

**Pf** ($\Rightarrow$) ✓

($\Leftarrow$) Suppose $f(\alpha)=0$ with $\alpha$ in some radical extension of $F$. Then $F(\alpha)/F$ solvable, so its Galois closure $M/F$ is solvable. By normality of $M/F$, $M$ contains the splitting field of $f$ over $F$ so $f$ is solvable by radicals. $\square$

**Recall** For $f \in F[x]$, $\mathrm{Gal}(f/F) = \mathrm{Gal}(L/F)$ for $L$ a splitting field of $f/F$.

**Thm** A polynomial $f \in F[x]$ is solvable by radicals$/_F$ iff $\mathrm{Gal}(f/F)$ is solvable. $\square$

**Prop** If $f \in F[x]$ has degree $n \leq 4$, then $f$ is solvable by radicals.

**Pf** If $f$ is separable, then $\mathrm{Gal}(f/F) \leq \Sigma_4$ which is solvable. For the nonseparable case, work with nonrepeated irred factors of $f$. $\square$

e.g. $\mathrm{Gal}(\underbrace{x^5-6x+3}_{\text{irreducible, so no root expressible by radicals!}}/\mathbb{Q}) \cong \Sigma_5$, not solvable.

- The Universal Polynomial:

$$\tilde{f} = x^2 - \sigma_1 x + \sigma_2 = (x-x_1)(x-x_2)$$

is solvable by radicals by the quadratic fmla.

Degree n generalization:

$$\tilde{f} = x^n - \sigma_1 x^{n-1} \cdots + (-1)^n \sigma_n = (x-x_1)\cdots(x-x_n)$$

solvable by radicals iff $L = F(x_1, \dots, x_n) / F(\sigma_1, \dots, \sigma_n) = K$ solvable iff $Gal(L/K) \cong \Sigma_n$ solvable. Hence have generic fmlae for roots iff $n \leq 4$.

Note Some polynomials of degree $>4$ are solvable by radicals.

- Abelian Equations:

Defn Let $f \in F[x]$. Call $f=0$ an Abelian equation if $f$ separable with root $\alpha$ s.t. the roots of $f$ are $\theta_1(\alpha), \dots, \theta_n(\alpha)$ for $\theta_1, \dots, \theta_n$ rational fns with coeffs in $F$ satisfying

$$\theta_i(\theta_j(\alpha)) = \theta_j(\theta_i(\alpha)) \quad \forall i,j.$$

Thm Let $f \in F[x]$. If $f=0$ is an Abelian equation, then $f$ is solvable by radicals over $F$.

Pf Abelian groups are solvable, so suffices to show $Gal(L/F)$ Abelian for $L$ splitting field of $f/F$. For $\sigma, \tau \in Gal(L/F)$, check that

- $\sigma(\alpha) = \theta_i(\alpha)$, $\tau(\alpha) = \theta_j(\alpha)$ for some $i, j$.
- $\sigma\tau = \tau\sigma$ iff $\sigma(\tau(\alpha)) = \tau(\sigma(\alpha))$
- $\sigma(\tau(\alpha)) = \theta_j(\theta_i(\alpha))$ and $\tau(\sigma(\alpha)) = \theta_i(\theta_j(\alpha))$. $\square$

**Thm** Let $f \in F[x]$ be irred and separable of degree $n$ with splitting field $L/F$. Then

$f=0$ is Abelian iff $Gal(L/F)$ is Abelian.

When these conditions are satisfied, $|Gal(L/F)| = [L:F] = n$ and $L = F(\alpha)$ for any root $\alpha \in L$ of $f$.

Pf Just saw $\Rightarrow$. For $\Leftarrow$, let $\alpha \in L$ be a root of $f$. Then

$$L / F(\alpha) / F \longleftrightarrow Gal(L/F(\alpha)) \underset{\text{since}}{\trianglelefteq} Gal(L/F)$$

since ↑ Abelian.

Thus $F(\alpha)/F$ is Galois, so $f$ splits completely in $F(\alpha)$ by normality. Thus $L = F(\alpha)$ and $[L:F] = n$. Each root is thus of the form $\theta_i(\alpha)$ for $\theta_i \in F(x)$. $\square$

**Reading** Thm 8.5.9: Artin's elegant proof of FTA.

It works for any extn $C/R$ where $R$ has no extns of odd degree $>1$, $C$ has no extns of deg 2.