Simple Groups

Defn A group G is *simple* if its only normal subgroups are 1 and G.

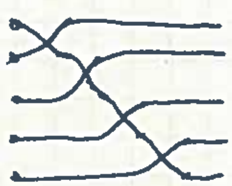
e.g. C_p for p prime (Lagrange's Thm)

Thm A_n is simple for $n \geq 5$.

Pf Two facts: ① l-cycle $(i_1 \cdots i_l) \in A_n$ iff l is odd

② For $n \geq 3$, A_n is gen'd by 3-cycles (HW)

For ①, $(i_1 \cdots i_l) = (i_1\ i_l) \cdots (i_1\ i_3)(i_1\ i_2)$.



Now suppose $H \neq 1 \trianglelefteq A_n$. Want to show $H = A_n$. First show H contains a 3-cycle. Take $1 \neq \sigma \in H$. Since $(j_1\ j_2\ j_3) \in A_n \trianglerighteq H$,

$$\sigma^{-1}(j_1\ j_2\ j_3)^{-1}\sigma\,(j_1\ j_2\ j_3) \in H.$$

If neither j nor $\sigma(j) \in \{j_1, j_2, j_3\}$, then $\sigma^{-1}(j_1\ j_2\ j_3)^{-1}\sigma(j_1\ j_2\ j_3)$ fixes j. Thus the elt in question moves at most 6 elts of $\{1, \ldots, n\}$.

Case 1 First suppose one of the cycles in σ has length ≥ 4, say $\sigma = (i_1\ i_2\ i_3\ i_4 \cdots)(\cdots)\cdots$. Then $\sigma^{-1}(i_2\ i_3\ i_4)^{-1}\sigma\,(i_2\ i_3\ i_4)$ $= (i_1\ i_3\ i_4)$. Indeed, fixes all $j \notin \{i_1, i_2, i_3, i_4\}$ and $i_2 \mapsto i_3 \mapsto i_4 \mapsto i_3 \mapsto i_2$. Etc.

Case 2 Suppose σ has a 3-cycle. If σ is a 3-cycle, we're done. So may assume $\sigma = (i_1\ i_2\ i_3)(i_4\ i_5 \cdots)\cdots$.

Then $\sigma^{-1}(i_2\ i_3\ i_5)^{-1}\sigma\,(i_2\ i_3\ i_5) = (i_1\ i_4\ i_2\ i_3\ i_5)$

so H contains a 5-cycle, so, by Case 1, H contains a 3-cycle.

<u>Case 3</u> Finally suppose σ is a product of disjoint 2-cycles $\sigma = (i_1\ i_2)(i_3\ i_4)\cdots$. Then $\sigma^{-1}(i_2\ i_3\ i_4)^{-1}\sigma\,(i_2\ i_3\ i_4)$ $= (i_1\ i_3)(i_2\ i_4) \in H$. Let i_5 be distinct from $i_1, \dots, i_4$ (using $n \geq 5$). Then

$$((i_1\ i_3)(i_2\ i_4))^{-1}(i_1\ i_3\ i_5)^{-1}((i_1\ i_3)(i_2\ i_4))(i_1\ i_3\ i_5)$$

$$= (i_1\ i_5\ i_3) \in H.$$

Now know some $(i\ j\ k) \in H$ and want to show all 3-cycles $\in H$. Suppose i', j', k' distinct, and let $\theta \in \Sigma_n$ satisfy

$$\theta(i) = i', \quad \theta(j) = j', \quad \theta(k) = k'.$$

Then $\theta(i\ j\ k)\theta^{-1} = (i'\ j'\ k')$. If $\theta \in A_n$, get $(i'\ j'\ k') \in H \trianglelefteq A_n$. If $\theta \notin A_n$, then $\theta' = \theta(i\ j) \in A_n$ and $\theta'(i\ j\ k)\theta'^{-1} = (j'\ i'\ k') \in H$ so $(i'\ j'\ k') = (j'\ i'\ k')^{-1} \in H$.

As H contains all 3-cycles, $H = A_n$. $\square$

<u>Lemma</u> Let G be a nonAbelian finite simple group. Then G is not solvable.

<u>Pf</u> Suppose $\cdots \trianglelefteq G_1 \trianglelefteq G_0 = G$ witnesses solvability. Then $G_1 = 1$ by simplicity of G and $[G:G_1] = |G| = p$, prime. But then $G = C_p$ is Abelian. $\square$

<u>Thm</u> A_n, Σ_n solvable iff $n \leq 4$.