

Solvable Groups

Defn A finite group  $G$  is solvable if there are subgroups

$$1 = G_n \trianglelefteq G_{n-1} \trianglelefteq \dots \trianglelefteq G_1 \trianglelefteq G_0 = G$$

s.t. for  $i=1, \dots, n$  we have

(a)  $G_i \trianglelefteq G_{i-1}$

(b)  $[G_{i-1} : G_i]$  is prime. (so  $G_i / G_{i-1} \cong C_p$ )

eg. The chain  $1 \trianglelefteq A_3 \trianglelefteq \Sigma_3$  exhibits  $\Sigma_3$  as solvable.

• All finite abelian groups are solvable (soon).

•  $A_n, \Sigma_n$  are nonsolvable for  $n \geq 5$  (later).

Prop Every subgp of a finite solvable gp is solvable.

pf Let  $\{G_i\}_{i=0}^n$  be a chain witnessing solvability of  $G$ .

For  $H \trianglelefteq G$  define  $H_i = H \cap G_i$  and note  $H_0 = H \cap G_0 = H \cap G = H$   
 $H_n = H \cap 1 = 1$ .

Let  $\pi$  be the composite  $H_{i-1} \hookrightarrow G_{i-1} \rightarrow G_{i-1}/G_i$ .

$$\text{Then } \ker \pi = \{h \in H_{i-1} \mid hG_i = G_i\}$$

$$= H_{i-1} \cap G_i = (H \cap G_{i-1}) \cap G_i$$

$$= H \cap G_i = H_i \trianglelefteq H_{i-1}$$

By the first isomorphism thm,

$$H_{i-1} / H_i \cong \text{im}(\pi) \leq G_{i-1} / G_i$$

$$\text{so } H_{i-1} / H_i \cong 1 \text{ or } C_p.$$

$\Downarrow$

$$H_i = H_{i-1}$$

So discarding duplicates we get a chain witnessing solvability of  $H$ .  $\square$

Thm  $H \trianglelefteq G$  finite. Then  $G$  is solvable iff  $H$  and  $G/H$  are solvable.

Pf First suppose  $G$  solvable. Then  $H$  is solvable by the prop. Let  $\pi: G \rightarrow G/H$  be the quotient hom. and set  $\tilde{G}_i = \pi(G_i)$ . Exc After discarding duplicates,  $\tilde{G}_i$  give a chain witnessing solvability of  $G/H$ .  
 Now suppose  $H, G/H$  solvable with

$$1 = H_e \leq H_{e-1} \leq \dots \leq H_0 = H$$

$$1 = \tilde{G}_m \leq \dots \leq \tilde{G}_0 = G/H$$

witnessing solvability. Then

$$1 = H_e \leq \dots \leq H_0 = H \leq \pi^{-1}\tilde{G}_m \leq \dots \leq \pi^{-1}\tilde{G}_0 = G$$

witnesses solvability of  $G$ . (check).  $\square$

Prop Every finite abelian group  $G$  is solvable.

Pf by strong induction on  $n = |G|$ . The case  $n=1$  is trivial. Assume  $G$  abelian of order  $n > 1$  and the result is true  $\forall$  abelian gps of order  $< n$ .

Let  $p$  be a prime divisor of  $n$ . If  $p=n$ ,  $G \cong C_p$  solvable. If  $p < n$ , Cauchy's thm says there is  $\langle g \rangle \leq G$ ,  $\langle g \rangle \cong C_p$ . This is solvable & normal since  $G$  abelian.  $|G/\langle g \rangle| < n$  so  $G/\langle g \rangle$  solvable, so the prop follows from the theorem.  $\square$ .

e.g.  $\mathbb{F}_p \cong T \trianglelefteq \text{AGL}(\mathbb{F}_p)$  with  $\text{AGL}(\mathbb{F}_p)/T \cong \mathbb{F}_p^\times$ .

Both  $\mathbb{F}_p, \mathbb{F}_p^\times$  abelian, hence solvable, so  $\text{AGL}(\mathbb{F}_p)$  is solvable.

Book Feit-Thompson theorem: Every gp of odd order is solvable. Pf 255pp.  $\square$

## Radical & Solvable Extensions

Defn A field extension  $L/F$  is radical if there are fields

$$F = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = L \text{ where for } i=1, \dots, n \exists \delta_i \in F_i \text{ s.t.} \\ F_i = F_{i-1}(\delta_i) \text{ and } \delta_i^{m_i} \in F_{i-1} \text{ for some integer } m_i > 0.$$

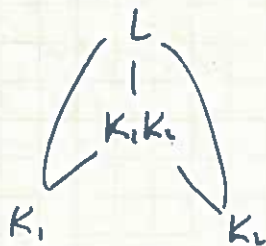
Note if  $b_i = \delta_i^{m_i}$  then  $F_i = F_{i-1}(\sqrt[m_i]{b_i})$ , i.e. radical extns arise by adjoining successive radicals.

e.g.  $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2})(\sqrt{2+\sqrt{2}}) = \mathbb{Q}(\sqrt{2+\sqrt{2}})$   
witnesses  $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$  as a radical extn.

Defn A field extn  $L/F$  is solvable (by radicals) if there is a field extn  $M/\mathbb{C}$  s.t.  $M/F$  is radical.

e.g. The splitting field of  $x^3 + x^2 - 2x + 1 / \mathbb{Q}$  is solvable but not radical.

Defn Suppose  $K_1, K_2 \subseteq L$  subfields. The composition  $K_1 K_2$  of  $K_1$  &  $K_2$  is the smallest subfield of  $L$  containing  $K_1, K_2$ .



Existence: Fields are closed under arbitrary intersection.

Prop  $M/\mathbb{C}/F$  with  $M/\mathbb{C}$  Galois. Then the composition of all conjugate fields of  $L$  in  $M$  is the Galois closure of  $L/F$ .

Lemma  $M/L_1, L_2/F$  with  $M/\mathbb{C}$  Galois, then

$$\cancel{\text{Gal}(M/L_1 L_2 / F)} = \cancel{\text{Gal}(L_1 / F) \cap \text{Gal}(L_2 / F)} \\ \text{Gal}(M/L_1 L_2) = \text{Gal}(M/L_1) \cap \text{Gal}(M/L_2).$$



Pf Lemma If  $\sigma$  fixes  $L_1, L_2$  then it fixes  $L_1, L_2$  so

$$\text{Gal}(M/L, L_2) \subseteq \text{Gal}(M/L_1) \cap \text{Gal}(M/L_2)$$

Suppose  $\sigma \in \text{Gal}(M/L_1) \cap \text{Gal}(M/L_2)$  suppose for  $\mathcal{Q}$  that

$\sigma x \neq x$  for some  $x \in L_1, L_2$ . Then  $M^{\langle \sigma \rangle} \cap L_1, L_2 \not\subseteq L_1, L_2$

with  $L_1, L_2 \subseteq M^{\langle \sigma \rangle} \cap L_1, L_2, \mathcal{Q}$ .  $\square$

Pf Prop Composition of the  $\sigma L$ ,  $\sigma \in \text{Gal}(M/F)$  has Galois

gp  $\bigcap_{\sigma \in \text{Gal}(M/F)} \sigma \text{Gal}(M/L) \sigma^{-1}$ , which is clearly normal in  $\text{Gal}(M/F)$

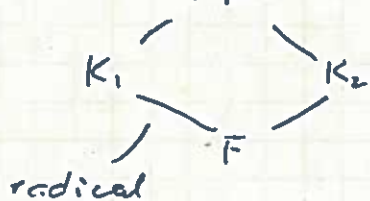
so  $\text{Conj}(\sigma L) / F$  is Galois and contains all ~~Galois~~

~~conjugates~~ of  $L$ . Now check that any Galois extn containing  $L$  contains all  $\sigma L$  (etc).  $\square$

Properties of radical & solvable extns

Lemma (a) If  $L/F, M/L$  are radical, so is  $M/F$ .

(b) 
$$\begin{array}{c} L \\ | \\ K_1 K_2 \end{array} \Rightarrow K_1 K_2 / K_2 \text{ radical.}$$



(c)  $K_1/F, K_2/F$  radical  $\Rightarrow K_1 K_2/F$  radical

Pf (a) follows from defs & (c)  $\Leftarrow$  (b).

For (b), the idea is to adjoin the same roots to  $K_2$  (check details).  $\square$

Thm If  $L/F$  is separable and radical, then the Galois closure of  $L$  is also radical.

Pf The Galois conjugates of  $L$  are radical.  $\square$

Cor Solvable extns of char 0 fields have solvable Galois closure.  $\square$