The Discriminant

For a nonconstant monic $f \in F[x]$, have discriminant $\Delta(f) \in F$.

If $n = \deg(f) \geq 2$ and $f = (x-\alpha_1)\cdots(x-\alpha_n)$ in a splitting field L of f,

then $\Delta(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2$ and f is seperable iff $\Delta(f) \neq 0$.

Define $\sqrt{\Delta(f)} = \prod_{i<j} (\alpha_i - \alpha_j) \in L$.

Recall that for f seperable, the action of $\mathrm{Gal}(L/F)$ on roots $\{\alpha_1, \dots, \alpha_n\}$ determines $\mathrm{Gal}(L/F) \hookrightarrow \Sigma_n$.

Thm Let f, L/F be as above and assume char $F \neq 2$.

(a) If $\sigma \in \mathrm{Gal}(L/F) \longmapsto \tau \in \Sigma_n$, then

$$\sigma\left(\sqrt{\Delta(f)}\right) = \mathrm{sgn}(\tau)\sqrt{\Delta(f)}.$$

(b) The image of $\mathrm{Gal}(L/F)$ lies in the alternating group A_n iff $\sqrt{\Delta(f)} \in F$ (i.e. $\Delta(f) = a^2$ for some $a \in F$).

Pf Recall $\sqrt{\Delta} = \prod_{i<j}(x_i - x_j) \in F[x_1, \dots, x_n]$ has the property

$$\tau\sqrt{\Delta} = \mathrm{sgn}(\tau)\sqrt{\Delta} \text{ for } \tau \in \Sigma_n.$$

Eval'n at $x_1 = \alpha_1, \dots, x_n = \alpha_n$ gives

$$\prod_{i<j}(\alpha_{\tau(i)} - \alpha_{\tau(j)}) = \mathrm{sgn}(\tau)\prod_{i<j}(\alpha_i - \alpha_j) = \mathrm{sgn}(\tau)\sqrt{\Delta(f)}$$

but $\sigma(\alpha_i) = \alpha_{\tau(i)}$ by defn, so the LHS $= \sigma\left(\sqrt{\Delta(f)}\right)$. Thus (a).

For (b), L/F is Galois, so $F = L^{\mathrm{Gal}(L/F)}$. Thus

$$\sqrt{\Delta(f)} \in F \iff \sigma\left(\sqrt{\Delta(f)}\right) = \sqrt{\Delta(f)} \ \forall \sigma \in \mathrm{Gal}(L/F)$$

$$\iff \mathrm{sgn}(\tau)\sqrt{\Delta(f)} = \sqrt{\Delta(f)} \ \forall \sigma$$

$$\iff \mathrm{sgn}(\tau) = 1 \ \forall \sigma. \qquad \square$$

<u>Prop</u> Let $f \in F[x]$ be a monic irred sep cubic, char $F \neq 2$. If L is the splitting field of f over F, then

$$\mathrm{Gal}(L/F) \cong \begin{cases} C_3 & \text{if } \Delta(f) \text{ is a square in } F \\ \Sigma_3 & \text{o/w}. \end{cases}$$

<u>Pf</u> For α a root of F, $L/F(\alpha)/F$ and $[F(\alpha):F]=3$, so $[L:F]$ is a multiple of 3. We also have $\mathrm{Gal}(L/F) \hookrightarrow \Sigma_3$ and the only subgps of Σ_3 of order divisible by 3 are Σ_3 and $A_3 \cong C_3$. $\square$

<u>The Universal Extension</u>

$L = F(x_1, \ldots, x_n) / K = F(\sigma_1, \ldots, \sigma_n)$ for σ_i the elementary symm polys.

From reading: L is the splitting field of

$$\tilde{f} = x^n - \sigma_1 x^{n-1} + \cdots + (-1)^n \sigma_n = \prod_{i=1}^{n} (x - x_i),$$

and $\mathrm{Gal}(L/K) \cong \Sigma_n$. Under this identification, $\sigma \in \Sigma_n$ permutes the x_i according to σ.

<u>Thm</u> Let $R \in F(x_1, \ldots, x_n)$ be a rat'l fn.

(a) R is invariant under Σ_n iff $R \in F(\sigma_1, \ldots, \sigma_n)$

(b) Assume char $F \neq 2$. Then R is invariant under A_n iff $\exists A, B \in F(\sigma_1, \ldots, \sigma_n)$ s.t. $R = A + B\sqrt{\Delta}$.

<u>Pf</u> (a) $L^{\mathrm{Gal}(L/K)} = K$.

(b) Let $M = L^{A_n}$. Since $[\Sigma_n : A_n] = 2$, $[M:K] = 2$.

Since $\tau\sqrt{\Delta} = \mathrm{sgn}(\tau)\sqrt{\Delta}$, $\sqrt{\Delta} \in M$, so $K \subseteq K(\sqrt{\Delta}) \subseteq M$.

Thus $2 = [M:K] = [M:K(\sqrt{\Delta})][K(\sqrt{\Delta}):K]$. But $\sqrt{\Delta} \notin K$ so $K(\sqrt{\Delta}) = M$.

$\square$