The splitting field of x^8-2

The splitting field of $x^8-2\,/\mathbb{Q}$ is gen'd by $\theta = \sqrt[8]{2} \in \mathbb{R}$ and $\zeta = \zeta_8 = e^{2\pi i/8}$.

Note that $i = \zeta_4 \in \mathbb{Q}(\zeta_8)$ and $\zeta_8 + \zeta_8^7 = \sqrt{2} \in \mathbb{Q}(\zeta_8)$
$\Rightarrow \mathbb{Q}(i,\sqrt{2}) \subseteq \mathbb{Q}(\zeta_8)$. In fact, $m_{\zeta,\mathbb{Q}} = x^4+1$
so $\mathbb{Q}(\zeta_8) = \mathbb{Q}(i,\sqrt{2})$.

Since $\theta^4 = \sqrt{2}$, get that sp. field of x^8-2 is gen'd by θ, i.
$[\mathbb{Q}(\theta):\mathbb{Q}] = 8$ b/c θ has min'l poly x^8-2 (irred by Eisenstein).
$\mathbb{Q}(\theta) \subseteq \mathbb{R}$ so $i \notin \mathbb{Q}(\theta)$ so

$$\begin{array}{c} \mathbb{Q}(\theta,\zeta) = \mathbb{Q}(\theta, i) \\ \Big| \, 2 \\ \mathbb{Q}(\theta) \\ \Big| \, 8 \\ \mathbb{Q} \end{array} \qquad 16$$

The Galois gp is determined by its action on θ, i:

$$\theta \longmapsto \zeta^a \theta \qquad a = 0,1,\dots,7$$
$$i \longmapsto \pm i$$

are possible, and there are only 16 of these, so they're all realized. Define

$$\sigma : \begin{cases} \theta \mapsto \zeta\theta \\ i \mapsto i \end{cases} \qquad \tau : \begin{cases} \theta \mapsto \theta \\ i \mapsto -i \end{cases}$$

Note that $\zeta = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1}{2}(1+i)\sqrt{2} = \frac{1}{2}(1+i)\theta^4$

Thus $\sigma(\zeta) = -\zeta = \zeta^5$, $\tau(\zeta) = \zeta^7$

Now compute:

f	$f(\theta)$	$f(i)$	$f(\zeta)$
σ	$\zeta\theta$	i	ζ^5
σ^2	$\zeta^6\theta$	i	ζ
σ^3	$\zeta^7\theta$	i	$-\zeta$
σ^4	$-\theta$	i	ζ
σ^5	$\zeta^5\theta$	i	$-\zeta$
σ^6	$\zeta^2\theta$	i	ζ
σ^7	$\zeta^3\theta$	i	$-\zeta$
τ	θ	$-i$	ζ^7
$\tau\sigma$	$\zeta^7\theta$	$-i$	ζ^3
$\tau\sigma^2$	$\zeta^2\theta$	$-i$	ζ^7
$\tau\sigma^3$	$\zeta\theta$	$-i$	ζ^3
$\tau\sigma^4$	$-\theta$	$-i$	ζ^7
$\tau\sigma^5$	$\zeta^3\theta$	$-i$	ζ^3
$\tau\sigma^6$	$\zeta^6\theta$	$-i$	ζ^7
$\tau\sigma^7$	$\zeta^5\theta$	$-i$	ζ^3

This exhausts the possibilities, (together with id) so σ, τ generate $\mathrm{Gal}(\mathbb{Q}(\theta,i)/\mathbb{Q})$.

Clearly $\tau^2 = 1$, $(\sigma^4)^2 = 1$ so $\sigma^8 = \tau^2 = 1$.

Also $\sigma\tau : \begin{cases} \theta \longmapsto \zeta\theta \\ i \longmapsto -i \\ \zeta \longmapsto \zeta^3 \end{cases}$

so $\sigma\tau = \tau\sigma^3$.

There are no other relns (why?) so

$\mathrm{Gal}(\mathbb{Q}(\theta,i)/\mathbb{Q})$

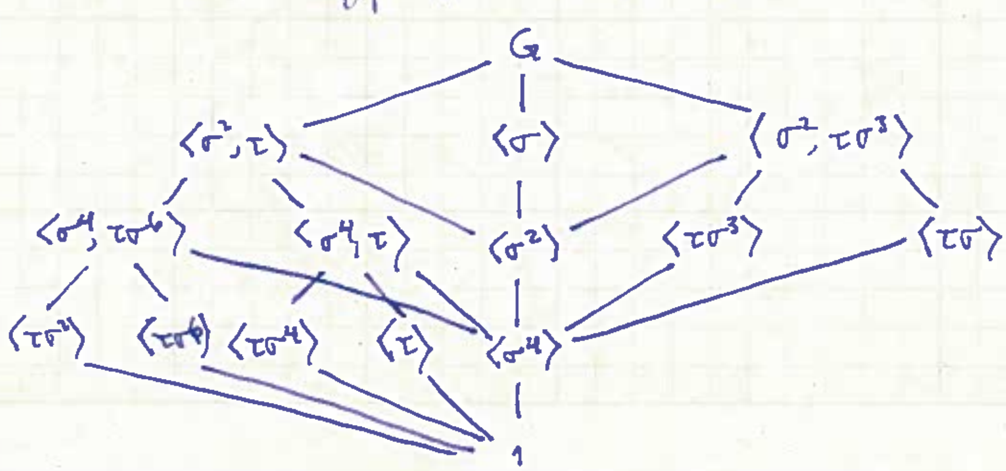
$= \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1,\ \sigma\tau = \tau\sigma^3 \rangle$

the quasidihedral group of order 16.

TPS Why can't θ, ζ be independently assigned?

A Algebraic dependence $\theta^4 = \sqrt{2} = \zeta + \zeta^7$.

Lattice of subgps of $G = \mathrm{Gal}(\mathbb{Q}(\theta,i)/\mathbb{Q})$:



What is the corresponding lattice of subextensions?

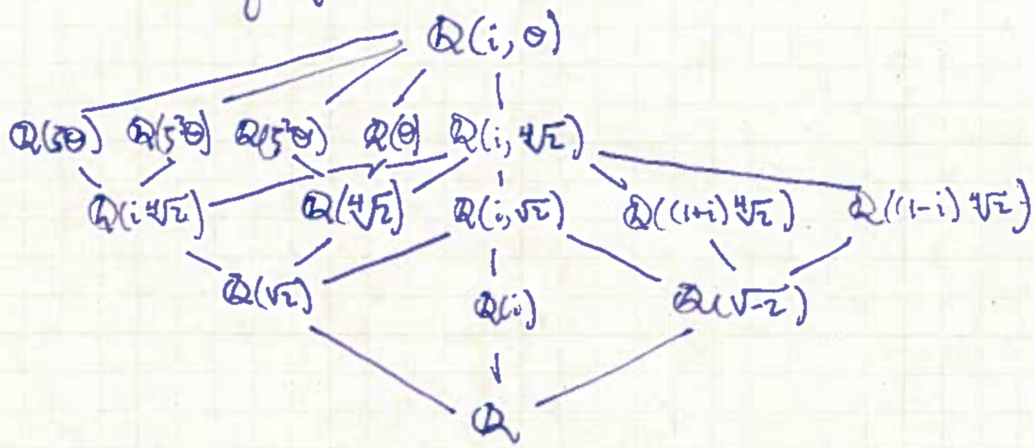
For $\mathbb{Q}(\theta, i) / K / \mathbb{Q}$ with $K = \mathbb{Q}(\theta, i)^H$,

$[K : \mathbb{Q}] = [G : H]$, so it suffices to find K of the correct degree fixed by (the generators of) H.

e.g. $\mathbb{Q}(i)$ is fixed by σ, $[G : \langle\sigma\rangle] = 2$, and $[\mathbb{Q}(i) : \mathbb{Q}] = 2$,

so $\mathbb{Q}(i) = \mathbb{Q}(\theta, i)^{\langle\sigma\rangle}$.

Ultimately get



e.g. $H = \langle\tau\sigma^3\rangle$. $\theta^2 = \sqrt[4]{2}$ fixed by σ^4, $\langle\sigma^4\rangle \trianglelefteq H$ of index 2 with coset reps $1, \tau\sigma^3$. Consider

$$\alpha = (1 + \tau\sigma^3)\theta^2 = \theta^2 + \tau\sigma^3\theta^2$$

$$\tau\sigma^3\alpha = (\tau\sigma^3 + (\tau\sigma^3)^2)\theta^2$$

$$= (\tau\sigma^3 + \sigma^4)\theta^2$$

$$= \alpha \quad \text{since } \sigma^4\theta^2 = \theta^2$$

Now $\alpha = \sqrt[4]{2} + i\sqrt[4]{2} = (1+i)\sqrt[4]{2} \in \mathbb{Q}(i, \theta)^H$.

Check $\sigma^2\alpha \neq \alpha$, so subgp diagram $\Rightarrow \mathbb{Q}(i,\theta)^H = \mathbb{Q}((1+i)\sqrt[4]{2})$.

Note $\tau H \tau^{-1} = \langle\tau\sigma\rangle$ has fixed field $\tau\mathbb{Q}(\alpha) = \mathbb{Q}(\tau\alpha) = \mathbb{Q}((1-i)\sqrt[4]{2})$.