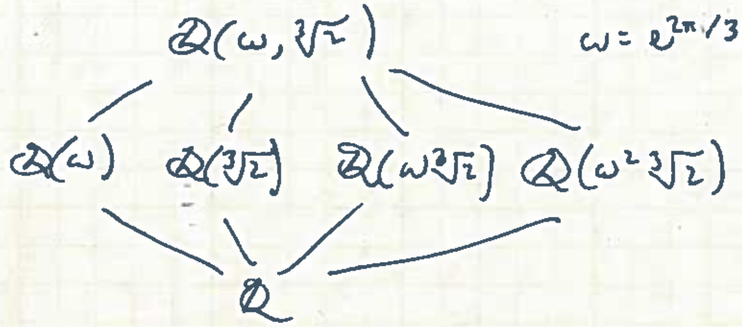
Normal Subgroups / Normal Extensions

A. Conjugate Fields

Defn For finite extns $L/K/F$, $\sigma \in \mathrm{Gal}(L/F)$, call
$$\sigma K = \{\sigma(\alpha) \mid \alpha \in K\}$$
a *conjugate field of* K.

Note $[K:F] = [\sigma K : F]$ b/c $K \xrightarrow[\cong]{\sigma} \sigma K$ over F.

e.g. $\omega = e^{2\pi i/3}$



$\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q})$ is determined by $\sigma(\omega) \in \{\omega, \omega^2\}$ and $\sigma(\sqrt[3]{2}) \in \{\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}\}$. It's easy to check that $\sigma\mathbb{Q}(\omega) = \mathbb{Q}(\omega)$ $\forall \sigma$. $\mathbb{Q}(\sqrt[3]{2})$ has $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(\omega\sqrt[3]{2})$, $\mathbb{Q}(\omega^2\sqrt[3]{2})$ as its conjugates.

Lemma Finite extns $L/K/F$. Then

(a) $\mathrm{Gal}(L/K) \leq \mathrm{Gal}(L/F)$

(b) If $\sigma \in \mathrm{Gal}(L/F)$, then $\mathrm{Gal}(L/\sigma K) = \sigma\,\mathrm{Gal}(L/K)\,\sigma^{-1}$ in $\mathrm{Gal}(L/F)$.

Pf (a) ✓ since $F \subseteq K$.

(b) Let $\gamma \in \sigma\,\mathrm{Gal}(L/K)\,\sigma^{-1}$, $\beta \in \sigma K$. Then $\gamma = \sigma\tau\sigma^{-1}$ for some $\tau \in \mathrm{Gal}(L/K)$, and $\beta = \sigma(\alpha)$ for some $\alpha \in K$. Thus
$$\gamma(\beta) = \sigma\tau\sigma^{-1}(\sigma(\alpha)) = \sigma\tau(\alpha) = \sigma(\alpha) = \beta$$
$\Rightarrow \gamma|_{\sigma K} = id \Rightarrow \sigma\,\mathrm{Gal}(L/K)\,\sigma^{-1} \leq \mathrm{Gal}(L/\sigma K)$.

$\geq$ similar. $\square$

B. Normal subgps

Thm Suppose $L/K/F$ where L/F Galois. Then TFAE:

(a) $K = \sigma K$ $\forall \sigma \in \mathrm{Gal}(L/F)$

(b) $\mathrm{Gal}(L/K) \trianglelefteq \mathrm{Gal}(L/F)$

(c) K/F Galois

(d) K/F normal.

Pf (a) $\Rightarrow$ (b): If $K = \sigma K$, then $\mathrm{Gal}(L/K) = \mathrm{Gal}(L/\sigma K) = \sigma \mathrm{Gal}(L/K)\sigma^{-1}$

so $\mathrm{Gal}(L/K) \trianglelefteq \mathrm{Gal}(L/F)$.

(b) $\Rightarrow$ (a): $\mathrm{Gal}(L/K) \underset{\substack{\uparrow \\ \text{normality}}}{=} \sigma \mathrm{Gal}(L/K)\sigma^{-1} = \mathrm{Gal}(L/\sigma K)$

L/K & $L/\sigma K$ Galois, so $K = L^{\mathrm{Gal}(L/K)} = L^{\mathrm{Gal}(L/\sigma K)} = \sigma K$.

(c) $\Rightarrow$ (d): ✓ as every Galois extn is normal and sep.

(d) $\Rightarrow$ (c): L/F Galois $\Rightarrow$ L/F sep $\Rightarrow$ K/F sep.

Thus K/F normal & sep, hence Galois.

(a) $\Rightarrow$ (d): Let $f \in F[x]$ be irred $/F$, root $\alpha \in K$. Then

$f = a_0 \prod_{i=1}^{r} (x - \alpha_i)$ for $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r \in L$ distinct elts of L obtained by applying elts of $\mathrm{Gal}(L/F)$ to α.

Since $\alpha \in K$, each $\alpha_i \in \sigma K = K \Rightarrow f$ splits completely over K.

(d) $\Rightarrow$ (a): Take $\alpha \in K$, $\sigma \in \mathrm{Gal}(L/F)$, and let $p = m_{\alpha,F}$.

Then $\sigma(\alpha)$ is also a root of p. Since K/F is normal, p splits completely over $K \Rightarrow \sigma(\alpha) \in K \Rightarrow \sigma K \subseteq K$.

Since these fields have the same degree over F, $\sigma K = K$. $\square$

cf. Example 7.2.6 in Cox to see the implications of this theorem for $\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q}$.

<u>Thm</u> Suppose $L/K/F$ with K/F & L/F Galois. Then $\mathrm{Gal}(L/K) \trianglelefteq \mathrm{Gal}(L/F)$ and $\mathrm{Gal}(L/F)/\mathrm{Gal}(L/K) \cong \mathrm{Gal}(K/F)$.

<u>Pf</u> If K/F Galois, then $\mathrm{Gal}(L/K) \trianglelefteq \mathrm{Gal}(L/F)$ by prev thm.
For fixed $\sigma \in \mathrm{Gal}(L/F)$, $\sigma|_K : K \cong \sigma K = K \Rightarrow \sigma|_K$ an aut of K/F.
Thus $\sigma \mapsto \sigma|_K$ defines $\Phi : \mathrm{Gal}(L/F) \to \mathrm{Gal}(K/F)$ which is clearly a homomorphism. Moreover,

$$\sigma \in \ker \Phi \iff \sigma|_K = \mathrm{id}_K \iff \sigma \in \mathrm{Gal}(L/K)$$

$\therefore \ker \Phi = \mathrm{Gal}(L/K)$. It remains to show $\mathrm{im}\,\Phi = \mathrm{Gal}(K/F)$.

But
$$|\mathrm{Im}\,\Phi| = |\mathrm{Gal}(L/F)/\mathrm{Gal}(L/K)| = \frac{[L:F]}{[L:K]} = [K:F] = |\mathrm{Gal}(K/F)|$$

$\therefore \mathrm{im}\,\Phi = \mathrm{Gal}(K/F)$. $\square$

<u>e.g.</u>
$$\begin{array}{l} L = \mathbb{Q}(\omega, \sqrt[3]{2}) \\ \ \big| \ \langle\sigma\rangle \\ \mathbb{Q}(\omega) \\ \ \big| \ \text{Galois} \\ \mathbb{Q} \end{array} \Longrightarrow \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \cong \mathrm{Gal}(L/\mathbb{Q})/\langle\sigma\rangle \cong \Sigma_3/A_3 \cong C_2.$$