Fundamental Thm of Galois Thy I

Let L/F be Galois.

(a) For $L/K/F$, $\mathrm{Gal}(L/K) \le \mathrm{Gal}(L/F)$ has fixed field

$$L^{\mathrm{Gal}(L/K)} = K.$$

Furthermore $|\mathrm{Gal}(L/K)| = [L:K]$ and $[\mathrm{Gal}(L/F):\mathrm{Gal}(L/K)] = [K:F]$.

(b) For $H \le \mathrm{Gal}(L/F)$, L^H has Galois gp

$$\mathrm{Gal}(L/L^H) = H.$$

Furthermore $[L:L^H] = |H|$ and $[L^H:F] = [\mathrm{Gal}(L/F):H]$.

Pf (a) L/K automatically Galois, so $L^{\mathrm{Gal}(L/K)} = K$.

$|\mathrm{Gal}(L/K)| = [L:K]$, $|\mathrm{Gal}(L/F)| = [L:F]$ since both extns Galois. Tower thm then gives

$$[\mathrm{Gal}(L/F):\mathrm{Gal}(L/K)] = \frac{[L:F]}{[L:K]} = [K:F].$$

(b) Take $H \le \mathrm{Gal}(L/F)$. Then $L/L^H/F$, and $H \le \mathrm{Gal}(L/L^H)$. L/L^H Galois, so

$$|H| \le |\mathrm{Gal}(L/L^H)| = [L:L^H]$$

Thus it suffices to show equality. Suppose for $\not\sim$ that $|H| < [L:L^H]$. Then $\exists\, \alpha_1, \dots, \alpha_{n+1} \in L$ which are L^H-lin ind. for $n = |H|$. Let $H = \{\overset{e}{\sigma_1}, \dots, \sigma_n\}$. Then the system

$$\begin{aligned} \sigma_1(\alpha_1)x_1 + \sigma_1(\alpha_2)x_2 + \cdots + \sigma_1(\alpha_{n+1})x_{n+1} &= 0 \\ &\vdots \\ \sigma_n(\alpha_1)x_1 + \sigma_n(\alpha_2)x_2 + \cdots + \sigma_n(\alpha_{n+1})x_{n+1} &= 0 \end{aligned} \qquad (*)$$

of n equations in $n+1$ unknowns $x_1, \dots, x_{n+1}$ has a solution $x_1 = \beta_1, \dots, x_{n+1} = \beta_{n+1}$ in L where not all $\beta_i = 0$. By lin ind of $\alpha_1, \dots, \alpha_{n+1}$ (and $\sigma_1 = e$) not all β_i are in L^H.

Among all nontrivial sol'ns $(\beta_1,\dots,\beta_{n+1})$ of ⊛, choose one with a minimal # of nonzero β_i. WLOG, $\beta_1,\dots,\beta_r \neq 0$, and dividing by β_r, $\beta_r = 1$. Know that at least 1 of $\beta_1,\dots,\beta_{r-1}, 1 \notin L^H$ (so $r > 1$), say $\beta_1 \notin L^H$. Then ⊛ becomes $\sigma_i(\alpha_1)\beta_1 + \cdots + \sigma_i(\alpha_{r-1})\beta_{r-1} + \sigma_i(\alpha_r) = 0$, $i = 1,\dots,n$.
Since $\beta_1 \notin L^H$, $\exists$ auto σ_{k_0} ($k_0 \in \{1,\dots,n\}$) with $\sigma_{k_0}\beta_1 \neq \beta_1$.
Applying σ_{k_0}, get

$$\sigma_{k_0}\sigma_i(\alpha_1)\,\sigma_{k_0}(\beta_1) + \cdots + \sigma_{k_0}\sigma_i(\alpha_{r-1})\,\sigma_{k_0}(\beta_{r-1}) + \sigma_{k_0}\sigma_i(\alpha_r) = 0$$

for $i = 1,\dots,n$. But $\{\sigma_{k_0}\sigma_i \mid i = 1,\dots,n\} = H = \{\sigma_1,\dots,\sigma_n\}$ so have

$$\sigma_i(\alpha_1)\,\sigma_{k_0}(\beta_1) + \cdots + \sigma_i(\alpha_{r-1})\,\sigma_{k_0}(\beta_{r-1}) + \sigma_i(\alpha_r) = 0$$

Subtracting systems, get

$$\sigma_i(\alpha_1)\big(\beta_1 - \sigma_{k_0}(\beta_1)\big) + \cdots + \sigma_i(\alpha_{r-1})\big(\beta_{r-1} - \sigma_{k_0}(\beta_{r-1})\big) = 0$$

for $i = 1,\dots,n$. This is a sol'n of ⊛ with fewer nonzero "β_i" and it's nontrivial since $\beta_1 \neq \sigma_{k_0}\beta_1$. ⚡

This proves $|H| = [L:L^H]$ and $\mathrm{Gal}(L/L^H) = H$.

$$\left.\begin{array}{c} L \\ \Big| \;|H| \\ L^H \\ \Big| \\ F \end{array}\right\} |\mathrm{Gal}(L/F)| \quad \Rightarrow \quad [L^H:F] = \frac{|\mathrm{Gal}(L/F)|}{|H|} = [\mathrm{Gal}(L/F):H]. \qquad \square$$

<u>FTGT II</u> L/F Galois. Then

$$\{K \mid L/K/F\} \xrightarrow{\ \sim\ } \{H \mid H \leq \mathrm{Gal}(L/F)\}$$
$$K \longmapsto \mathrm{Gal}(L/K)$$
$$L^H \longleftarrow\!\shortmid\ H$$

are inverses of each other which reverse inclusions.

Furthermore, if ~~K~~ $\mapsto H$ under this bij'n, then K/F is Galois iff $H \trianglelefteq \mathrm{Gal}(L/F)$, and when this happens, there is a natural isomorphism $\mathrm{Gal}(L/F)/H \cong \mathrm{Gal}(K/F)$.

Pf $K \longmapsto \mathrm{Gal}(L/K) \longmapsto L^{\mathrm{Gal}(L/K)} = K$ ✓

$H \longmapsto L^H \longmapsto \mathrm{Gal}(L/L^H) = H$ ✓

Inclusion-reversing is an easy check.

Normality portion proved Wednesday. $\square$