Algebraic Extensions

Defn A field extn L/F is algebraic if every element of L is algebraic over F.

Lemma Suppose L/F is finite. Then

(a) L/F is algebraic.

(b) If $\alpha \in L$, then $\deg(m_{\alpha,F}) \mid [L:F]$.

Pf For $\alpha \in L$, $F \subseteq F(\alpha) \subseteq L$ and the tower thm gives $[F(\alpha):F]$ finite, dividing $[L:F]$. We have already seen $[F(\alpha):F]$ finite $\Leftrightarrow$ α alg $/F$. $\square$

Note There are alg extns which are not finite.

Thm Let L/F be a field extn. Then $[L:F]<\infty$ iff $\exists \alpha_1,\dots,\alpha_m \in L$ s.t. each α_i is alg $/F$, and $L = F(\alpha_1,\dots,\alpha_m)$

Pf Suppose $[L:F]<\infty$ and take $\alpha_1,\dots,\alpha_m \in L$ a basis of L over F.
Then $L = \{a_1\alpha_1 + \dots + a_m\alpha_m \mid a_i \in F\} \subseteq F(\alpha_1,\dots,\alpha_m) \subseteq L$
so $L = F(\alpha_1,\dots,\alpha_m)$ and Lemma shows each α_i alg $/F$.
Now suppose $L = F(\alpha_1,\dots,\alpha_m)$ with each α_i alg $/F$.
Let $L_0 = F$, $L_i = F(\alpha_1,\dots,\alpha_i)$ for $1 \le i \le m$. Get $F = L_0 \subseteq L_1 \subseteq \dots \subseteq L_m = L$.
and $L_i = L_{i-1}(\alpha_i)$. Since α_i alg $/F$, it is also alg $/L_{i-1}$, so $[L_i : L_{i-1}] < \infty$. Thus $[L:F] = [L_m : L_{m-1}] \cdots [L_1 : L_0] < \infty$. $\square$

Prop Let L/F be a field extn. If $\alpha, \beta \in L$ alg $/F$, then $\alpha+\beta$, $\alpha\beta$ are alg $/F$ as well.

Pf By the thm, $F(\alpha,\beta)/F$ is a finite extn, hence algebraic. $\square$

Cor For any L/F, $M = \{\alpha \in L \mid \alpha \text{ alg}/F\}$ is a subfield of L containing F. $\square$

<u>Thm</u> Let $F \subseteq K \subseteq L$. If $\alpha \in L$ alg/K and K alg/F, then α alg/F.

<u>Pf</u> Let α be a root of $f = \beta_n x^n + \cdots + \beta_0 \in K[x]$ where $\beta_n, \ldots, \beta_0 \in K$, not all 0. Each β_i alg/F, so $M = F(\beta_n, \ldots, \beta_0)$ is a finite extn of F. Note $f \in M[x]$, so α alg/M, so $M(\alpha)/M$ is finite. Thus $[M(\alpha):F] = [M(\alpha):M][M:F] < \infty$, so α alg/F. $\square$

<u>e.g.</u> Every cpx soln of $x^{11} - (\sqrt{2} + \sqrt{5})x^5 + 3\sqrt[4]{12}\,x^3 + (1+3i)x + \sqrt[5]{17} = 0$ is an algebraic number.

<u>Cor</u> $L/K/F$ with L/K alg, K/F alg, then L/F algebraic.

<u>Defn</u> The algebraic #s $\overline{\mathbb{Q}} = \{z \in \mathbb{C} \mid z \text{ alg}/\mathbb{Q}\}$.

<u>Thm</u> The field $\overline{\mathbb{Q}}$ is algebraically closed.

<u>Pf</u> It suffices to show every nonconstant poly in $\overline{\mathbb{Q}}[x]$ has a root in $\overline{\mathbb{Q}}$. Given such f, it has a root $\alpha \in \mathbb{C}$. Thus α alg/$\overline{\mathbb{Q}}$ since it's a root of $f \in \overline{\mathbb{Q}}[x]$. By the corollary, α alg/$\mathbb{Q}$ so $\alpha \in \overline{\mathbb{Q}}$. $\square$