Degree

For any field extn L/F, L is an F-vector space.

Defn The degree of L/F is $[L:F] := \dim_F L$.

Call L/F a finite extension if $[L:F] < \infty$.

e.g.
- $[\mathbb{C}:\mathbb{R}] = 2$
- $[\mathbb{Q}(\sqrt{D}):\mathbb{Q}] = 2$ for D not a square in $\mathbb{Q}$.
- $[L:F] = 1$ iff $L = F$.

Prop $\alpha \in L/F$.

(a) α is alg $/F$ iff $[F(\alpha):F] < \infty$.

(b) Let α be alg $/F$. If $n =$ degree of min poly of α/F, then $1, \alpha, \dots, \alpha^{n-1}$ form a basis of $F(\alpha)$ over F. Thus $[F(\alpha):F] = n$.

Pf First suppose α alg $/F$ w/ min poly p, $n = \deg(p)$. Since $F(\alpha) = F[\alpha]$, every elt of $F(\alpha)$ is of the form $g(\alpha)$ for some $g \in F[x]$.

By the division algorithm, $g = qp + (a_0 + a_1 x + \cdots + a_{n-1} x^{n-1})$ w/ $q \in F[x]$, $a_i \in F$. Eval'n at $x = \alpha$ gives

$$g(\alpha) = a_0 + \cdots + a_{n-1}\alpha^{n-1}$$

Hence $1, \dots, \alpha^{n-1}$ span $F(\alpha)$ over F. Linear independence follows from minimality of $\deg(p)$. Thus $[F(\alpha):F] = n < \infty$.

Now suppose $[F(\alpha):F] = n < \infty$. Then $1, \alpha, \dots, \alpha^n$ are lin dep over F. Hence $\exists a_i \in F$ s.t. $a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0$. $\square$

e.g. Since min poly of $\sqrt{2}+\sqrt{3}$ $/\mathbb{Q}$ is $x^4 - 10x^2 + 1$,

$[\mathbb{Q}(\sqrt{2}+\sqrt{3}):\mathbb{Q}] = 4$ and every elt of $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ can be written uniquely in the form $a + b(\sqrt{2}+\sqrt{3}) + c(\sqrt{2}+\sqrt{3})^2 + d(\sqrt{2}+\sqrt{3})^3$, $a, b, c, d \in \mathbb{Q}$.

Towers

Thm Suppose we have fields $F \subseteq K \subseteq L$.

(a) If $[K:F] = \infty$ or $[L:K] = \infty$, then $[L:F] = \infty$.

(b) If $[K:F] < \infty$ and $[L:K] < \infty$, then $[L:F] = [L:K][K:F]$.

Diagrammatically:

$$\begin{tikzcd} L \arrow[dd, dash, "mn"'] \arrow[dr, dash, "n"] & \\ & K \arrow[dl, dash, "m"] \\ F & \end{tikzcd}$$

Pf (a) Suppose $[L:F] = N$ and let $\gamma_1, \dots, \gamma_N$ be a basis of L/F. Then K is an F-subspace of L, hence is finite dim'l $/F$, i.e. $[K:F] < \infty$. Take $\alpha \in L$. Then $\alpha = \sum_{i=1}^{N} a_i \gamma_i$ with $a_i \in F \subseteq K$, so L is spanned by $\gamma_1, \dots, \gamma_N$ as a K-vs. $\Rightarrow [L:K] \leq n < \infty$.

(b) Let $m = [K:F]$, $n = [L:K]$, and pick bases $\alpha_1, \dots, \alpha_m$ of K/F, $\beta_1, \dots, \beta_n$ of L/K. Show $\{\alpha_i \beta_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ are a basis of L/F: For $\gamma \in L$, $\gamma = \sum_{j=1}^{n} b_j \beta_j$, $b_j \in K$, $b_j = \sum_{i=1}^{m} a_{ij} \alpha_i$, $a_{ij} \in F$.

Thus $\gamma = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \alpha_i \beta_j$ so $\{\alpha_i \beta_j\}$ span L/F.

TBS Linear independence? $\square$

e.g. $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$.

Basis $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ of $\mathbb{Q}(\sqrt{2}, \sqrt{3}) / \mathbb{Q}$.

Note If we believe $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$, then

$$\begin{tikzcd} \mathbb{Q}(\sqrt{2}, \sqrt{3}) \arrow[dd, dash, "4"'] \arrow[dr, dash] & \\ & \mathbb{Q}(\sqrt{2} + \sqrt{3}) \arrow[dl, dash, "4"] \\ \mathbb{Q} & \end{tikzcd} \Rightarrow \mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

e.g. Let $\omega = e^{2\pi i/3}$.

$$\begin{tikzcd} \mathbb{Q}(\omega, \sqrt[3]{2}) \arrow[dd, dash, "6"'] \arrow[dr, dash, "2"] & \\ & \mathbb{Q}(\sqrt[3]{2}) \arrow[dl, dash, "3"] \\ \mathbb{Q} & \end{tikzcd}$$

2 b/c ω, ω^2 roots of $x^2 + x + 1$, $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$ so $x^2 + x + 1 = $ min poly of $\omega / \mathbb{Q}(\sqrt[3]{2})$

3 b/c $x^3 - 2$ irred by Eisenstein