Fundamental Theorem of Algebra Every nonconstant $f \in \mathbb{C}[x]$ splits completely over $\mathbb{C}$, i.e. $f = a_0(x-\alpha_1)\cdots(x-\alpha_n)$ for some $a_0, \alpha_1, \dots, \alpha_n \in \mathbb{C}$ with $a_0 \neq 0$.

Prop TFAE:

(a) Every nonconst $f \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$.

(b) Every nonconst $f \in \mathbb{C}[x]$ splits completely over $\mathbb{C}$.

(c) Every nonconst $f \in \mathbb{R}[x]$ has at least one root in $\mathbb{C}$.

Sketch (a) $\Rightarrow$ (b) by induction on degree.

(b) $\Rightarrow$ (c) is trivial since $\mathbb{R} \subseteq \mathbb{C}$.

For (c) $\Rightarrow$ (a), take $f = a_0x^n + \cdots + a_n \in \mathbb{C}[x]$. We must show that f has a root in $\mathbb{C}$ when $n>0$, $a_0 \neq 0$. Define $\bar{f} = \bar{a}_0x^n + \cdots + \bar{a}_n$. Check $\bar{f}\bar{g} = \overline{fg}$. Hence $\overline{f\bar{f}} = \bar{f}\bar{\bar{f}} = \bar{f}f = f\bar{f} \Rightarrow f\bar{f} \in \mathbb{R}[x]$.

By hypothesis, $\exists \alpha \in \mathbb{C}$ s.t. $(f\bar{f})(\alpha) = 0$. But then $f(\alpha)\bar{f}(\alpha) = 0$ so $f(\alpha) = 0$ or $\bar{f}(\alpha) = 0$. In the former case, $\alpha \in \mathbb{C}$ is a root of f; in the latter, $\bar{\alpha} \in \mathbb{C}$ is a root of f (check!). □

Prop Every $f \in \mathbb{R}[x]$ of odd degree has at least one root in $\mathbb{R}$.

Sketch WLOG, $f = x^n + a_1x^{n-1} + \cdots + a_n$ with n odd, $a_1, \dots, a_n \in \mathbb{R}$. For $x \gg 0$, $f(x) > 0$. For $x \ll 0$, $f(x) < 0$. Thus, by the intermediate value theorem (Math 112!), f has a root. □

Lemma Every quadratic polynomial in $\mathbb{C}[x]$ splits completely over $\mathbb{C}$.

Pf The roots of $f = ax^2 + bx + c$ with $a \neq 0$ are $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

$b^2 - 4ac = re^{i\theta}$ for some $r \geq 0 \in \mathbb{R}$. Hence

$\sqrt{b^2-4ac} = \sqrt{r}\, e^{i\theta/2} \in \mathbb{C}$ since $\sqrt{r}$ exists (again by IVT).

Hence the roots of f are in $\mathbb{C}$. □

Pf of FTA It suffices to show that every $f \in \mathbb{R}[x]$ of deg $n > 0$ has at least one root in $\mathbb{C}$. Write n as $n = 2^m k$, k odd, $m \geq 0$. We proceed by induction on m. If $m = 0$, $\deg(f) = k$ odd, so we're done by the Prop.

Now suppose ~~xxxxx~~ $m > 0$ and every $f \in \mathbb{R}[x]$ of degree $2^{m-1} \cdot (\text{odd})$ has at least one root in $\mathbb{C}$. $\exists L/\mathbb{C}$ s.t. f splits completely over L with roots $\alpha_1, \dots, \alpha_n \in L$.

Clever idea (Laplace): Set $\displaystyle g_\lambda(x) = \prod_{1 \leq i < j \leq n} (x - (\alpha_i + \alpha_j) + \lambda \alpha_i \alpha_j)$

where $\lambda \in \mathbb{R}$. $\deg(g_\lambda) = \frac{1}{2} n(n-1)$.

Claim $g_\lambda \in \mathbb{R}[x]$.

Justification Consider $\displaystyle G_\lambda(x) = \prod_{1 \leq i < j \leq n} (x - (x_i + x_j) + \lambda x_i x_j)$

G_λ is fixed by transpositions and hence by Σ_n. It follows that there are symmetric polynomials $p_i(x_1, \dots, x_n)$ s.t. $\displaystyle G_\lambda(x) = \sum_{i=0}^{\frac{1}{2}n(n-1)} p_i(x_1, \dots, x_n) x^i$. Since $\lambda \in \mathbb{R}$, $p_i \in \mathbb{R}[x_1, \dots, x_n]$.

By Cor 2.2.5, $p_i(\alpha_1, \dots, \alpha_n) \in \mathbb{R}$ since $\alpha_1, \dots, \alpha_n$ are the roots of $f \in \mathbb{R}[x]$. Thus $\displaystyle g_\lambda(x) = \sum_{i=0}^{\frac{1}{2}n(n-1)} p_i(\alpha_1, \dots, \alpha_n) x^i \in \mathbb{R}[x]$.

Now $\deg(g_\lambda) = \frac{1}{2} n(n-1) = \frac{1}{2} 2^m k (2^m k - 1) = 2^{m-1} \underbrace{k (2^m k - 1)}_{\text{odd}}$

Thus the induction hypothesis applies and g_λ has a root in $\mathbb{C}$. These roots are $\alpha_i + \alpha_j - \lambda \alpha_i \alpha_j$, so for each $\lambda \in \mathbb{R}$ we can find a pair i, j with $1 \leq i < j \leq n$ s.t. $\alpha_i + \alpha_j - \lambda \alpha_i \alpha_j \in \mathbb{C}$.

By the infinite $\to$ finite pigeonhole principle, $\exists \lambda \neq \mu \in \mathbb{R}$ and $1 \leq i < j \leq n$ s.t. $\alpha_i + \alpha_j - \lambda \alpha_i \alpha_j \in \mathbb{C}$ and $\alpha_i + \alpha_j - \mu \alpha_i \alpha_j \in \mathbb{C}$.

Subtracting, $(\mu-\lambda)\alpha_i\alpha_j \in \mathbb{C} \Rightarrow \alpha_i\alpha_j \in \mathbb{C} \Rightarrow \alpha_i+\alpha_j \in \mathbb{C}$.
Now consider the quadratic polynomial

$$(x-\alpha_i)(x-\alpha_j) = x^2-(\alpha_i+\alpha_j)+\alpha_i\alpha_j .$$

This has coeffs in $\mathbb{C}$ and hence roots in $\mathbb{C}$, so $\alpha_i, \alpha_j \in \mathbb{C}$. $\square$