Symmetric Polynomials

$G \curvearrowright S$ — group; (left) G-set

$S^G := \{s \in S \mid g \cdot s = s\}$ is the G-fixed set of S. (or G-invariants)

$\Sigma_n = S_n$ = permutations of $\{1, 2, \dots, n\}$ = symmetric group on n letters

$\Sigma_n \curvearrowright F[x_1, \dots, x_n]$ by permuting variables:

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Moral Exercise Check that this is an action: $e \cdot f = f$, $(\sigma\tau)f = \sigma(\tau f)$.

TPS $\sigma \cdot (f+g) = \sigma f + \sigma g$, $\sigma \cdot (fg) = (\sigma f)(\sigma g)$

and thus $F[x_1, \dots, x_n]^{\Sigma_n}$ is a ring.

Thm $F[x_1, \dots, x_n]^{\Sigma_n} = F[\sigma_1, \dots, \sigma_n]$, i.e., every symmetric polynomial is a polynomial in elementary symmetric polynomials. (and this expression is unique).

e.g. $x^3 + y^3 = (x+y)^3 - 3xy(x+y) = \sigma_1^3 - 3\sigma_1\sigma_2$.

Our proof uses graded lexicographic monomial order:

$$x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n} \iff a_1 + \cdots + a_n < b_1 + \cdots + b_n$$

or $\Sigma a_i = \Sigma b_i$ & $a_1 < b_1$

or $\Sigma a_i = \Sigma b_i$, $a_1 = b_1$, & $a_2 < b_2$

or $\Sigma a_i = \Sigma b_i$, $a_1 = b_1$, $a_2 = b_2$, & $a_3 < b_3$

or ...

e.g. $x_1^4 x_2^2 x_3 < x_1^2 x_2^3 x_3^3$, $x_1^4 x_2^2 x_3 > x_1^4 x_2 x_3^2$.

TPS Fix a monomial $x_1^{a_1} \cdots x_n^{a_n}$. Show that $\{x_1^{b_1} \cdots x_n^{b_n} < x_1^{a_1} \cdots x_n^{a_n}\}$ is finite.

Defn The (graded lexicographic) leading term of $f \neq 0 \in F[x_1, \dots, x_n]$ is the term of f with largest monomial in the grlex order.

Pf of Thm Take $f \in F[x_1, \dots, x_n]^{\Sigma_n}$, $f \neq 0$, with leading term $c x_1^{a_1} \cdots x_n^{a_n}$. By symmetry, $a_1 \geq a_2 \geq \cdots \geq a_n$ (check this!).

Set $g = \sigma_1^{a_1-a_2}\sigma_2^{a_2-a_3}\cdots\sigma_{n-1}^{a_{n-1}-a_n}\sigma_n^{a_n}$ and check that the leading term of g is $x_1^{a_1}\cdots x_n^{a_n}$. Hence $f_1 = f - cg$ has a strictly smaller leading term and is also symmetric.

Repeat this process to produce $f_2 = f_1 - c_1 g_1 = f - cg - c_1 g_1$, $f_3 = f - cg - c_1 g_1 - c_2 g_2$, etc. with $c_i \in F^\times$, g_i polynomials in $\sigma_1, \dots, \sigma_n$. At each stage, the leading term gets strictly smaller.

<u>TPS</u> Why does this process terminate with some $f_m = 0$?

If $f_m = f - cg - c_1 g_1 - \cdots - c_{m-1} g_{m-1} = 0$, then

$$f = cg + c_1 g_1 + \cdots + c_{m-1} g_{m-1}.$$

Uniqueness: Read the proof of Thm 2.2.7 in the textbook. $\square$

<u>Note</u> Uniqueness tells us $\sigma_1, \dots, \sigma_n$ are <u>algebraically independent</u>.

Write $\Sigma_n x_1^{a_1}\cdots x_n^{a_n} := \sum \Sigma_n \cdot \{x_1^{a_1}\cdots x_n^{a_n}\}$ so that

$\Sigma_2 x_1^2 x_2 = x_1^2 x_2 + x_2^2 x_1$ — add together everything in the Σ_n-orbit.

$\Sigma_3 x_1^2 x_2 = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2.$