The discriminant

For $n \geq 2$ variables $x_1, \dots, x_n$ over a field F, the discriminant is
$$\Delta := \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \in F[x_1, \dots, x_n].$$
$$= \left(\prod_{\substack{i \neq j \\ 1 \leq i, j \leq n}} (x_i - x_j) \right) \cdot (-1)^{\binom{n}{2}} \in F[x_1, \dots, x_n]^{\Sigma_n}.$$

Taking square root:
$$\sqrt{\Delta} = \prod_{1 \leq i < j \leq n} (x_i - x_j) \in F[x_1, \dots, x_n]$$

Prop For $\sigma \in \Sigma_n$, $\sigma \cdot \sqrt{\Delta} = \operatorname{sgn}(\sigma) \sqrt{\Delta}$

Pf HW! $\square$

Now define the discriminant of a polynomial $f = x^n + a_1 x^{n-1} + \dots + a_n \in F[x]$.
Let $\tilde{f} = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \dots + (-1)^n \sigma_n \in F[x, x_1, \dots, x_n]$.
Then $\tilde{f} \mapsto f$ under the map taking σ_i to $(-1)^i a_i$ (evaluation on $F[x, \sigma_1, \dots, \sigma_n]$).

Defn $\Delta(f) = \Delta(-a_1, a_2, \dots, (-1)^n a_n)$ where $\Delta = \Delta(\sigma_1, \dots, \sigma_n)$.
$\Delta(f) := 1$ if f has degree 1.

e.g. $f = x^2 + bx + c$
$\Delta = x_1^2 - 2x_1 x_2 + x_2^2 = \sigma_1^2 - 4\sigma_2$
$\Rightarrow \Delta(f) = b^2 - 4c$.

Prop If $f \in F[x]$ monic of deg $n \geq 2$ has roots $\alpha_1, \dots, \alpha_n$ in $L \supseteq F$, then $\Delta(f) = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$.

Pf Consider the evaluation map $x_i \mapsto \alpha_i$; then $\Delta \mapsto \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$.
If $\Delta = \Delta(\sigma_1, \dots, \sigma_n)$, then $x_i \mapsto \alpha_i$ takes Δ to
$\Delta(\sigma_1(\alpha_1, \dots, \alpha_n), \dots, \sigma_n(\alpha_1, \dots, \alpha_n)) = \Delta(-a_1, a_2, \dots, (-1)^n a_n) = \Delta(f)$.
$\square$

Note Let $R = F[x_1, \dots, x_n]$ and $A_n = \ker(\mathrm{sgn}) \leq \Sigma_n$ denote the alternating group. Then $R^{\Sigma_n} \subseteq R^{A_n} \subseteq R$ and $\sqrt{\Delta}$ is an example of an element of $R^{A_n} \setminus R^{\Sigma_n}$. In fact,

$$R^{A_n} = R^{\Sigma_n}[\sqrt{\Delta}] \Big/ ((\sqrt{\Delta})^2 - \Delta) = F[\sigma_1, \dots, \sigma_n, \sqrt{\Delta}] \Big/ (\sqrt{\Delta}^2 - \Delta).$$

We'll prove a function field version of this in Ch. 7.

Prop
$$\sqrt{\Delta} = \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \cdot (-1)^{n(n-1)/2}$$

Pf Call the matrix in question V. By the Leibniz (permutation) expansion of $\det V$,

$$\det V = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}.$$

Thus each term has degree $0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2}$.

If we set x_j equal to x_i $(i \neq j)$, V has two identical rows and thus 0 determinant. Thus $x_j - x_i$ is a factor of $\det V$. Hence $\det V = g \cdot \sqrt{\Delta}$ for some polynomial g. Clearly ~~det~~ $\sqrt{\Delta}$ is homogeneous of degree $\frac{n(n-1)}{2}$ so g is constant.

The $\sigma = e$ contribution to $\det V$ is $x_2 x_3^2 \cdots x_n^{n-1}$ which equals the summand of $\sqrt{\Delta}$ given by multiplying all first terms in $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)(x_4 - x_3)(x_4 - x_2)(x_4 - x_3) \cdots$.

Hence $g = 1$ and $\sqrt{\Delta} = \det V$. $\square$

(!) As written, proof neglects the sign — spot the mistake!