Formally Real Fields

Defn A field F is *formally real* if -1 is not a sum of squares in F; otherwise, F is called *nonreal*.

Notation $F^{\square} := \{a^2 \mid a \in F\}$

$F^{\blacksquare} := \{a^2 \mid a \in F^{\times}\} = F^{\square} \setminus \{0\}$.

$\sigma(F) = \{\sum_{i=1}^{n} a_i^2 \mid a_i \in F, n \in \mathbb{N}\}$

$\dot{\sigma}(F) = \sigma(F) \setminus \{0\}$

Note Formally real fields have char 0 b/c $\sigma(\mathbb{F}_p) = \mathbb{F}_p$ (check).

Prop (a) $\dot{\sigma}(F) \leq F^{\times}$

(b) If F is nonreal and char $F \neq 2$, then $\sigma(F) = F$.

Note If char $F = 2$, $\sigma(F) = F^{\square}$.

Pf (a) Easy to check closure of $\dot{\sigma}(F)$ under mult'n.

If $0 \neq a = a_1^2 + \cdots + a_n^2 \in F$, then

$$\frac{1}{a} = \frac{a}{a^2} = \left(\frac{a_1}{a}\right)^2 + \cdots + \left(\frac{a_n}{a}\right)^2 \in \dot{\sigma}(F).$$

(b) Given $x \in F$, we have $x = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 \in F^{\square} + \sigma(F)F^{\square} \subseteq \sigma(F)$. $\square$

Defn An *ordering* on F is a set $P \subsetneq F$ called the *positive cone* of the ordering s.t.

(1) $P + P \subseteq P$

(2) $P \cdot P \subseteq P$

(3) $P \cup (-P) = F$.

Prop Let (F, P) be any ordered field. Then

(1) $\sigma(F) \subseteq P$

(2) $-1 \notin P$, and $P \cap (-P) = \{0\}$

(3) F is formally real

(4) $P^{\times} := P \setminus \{0\}$ is a subgp of index 2 in $F^{\times}$.

(5) If $P' \subsetneq F$ is another ordering, $P \subseteq P' \Rightarrow P = P'$

Pf Moral exc. Note (2) follows from same trick as (b) above, and (2) $\Rightarrow$ (3). $\square$

Note • $F = P^\times \sqcup \{0\} \sqcup (-P^\times)$ so we can define a relation $\leq_P$ on F by $x \leq_P y$ iff $y - x \in P$. Get that $\leq_P$ is a total ordering on F.

• For F/F_0 and $P \subsetneq F$ an ordering, get an induced ordering $P_0 := F_0 \cap P$ on F_0

• $\mathbb{R}$ has a unique ordering by $\mathbb{R}^\square = \sigma(\mathbb{R}) = \mathbb{R}_{\geq 0}$.

Lemma Let F be formally real and $K = F(\sqrt{a})$ be a quadratic extn of F. Then K is nonreal iff $a \in \dot{\sigma}(F)$.

Pf If $-a \in \dot{\sigma}(F)$, then $(\sqrt{a})^2 + (-a) = 0$ shows that K is nonreal.

Conversely, if K is nonreal, have $-1 = \sum (b_i + c_i\sqrt{a})^2$, $b_i, c_i \in F$.

So $-1 = \sum b_i^2 + a \sum c_i^2$. Now $\sum c_i^2 \neq 0$ (o/w $-1 = \sum b_i^2 \in \sigma(F)$)

so $-a = \dfrac{1 + \sum b_i^2}{\sum c_i^2} \in \dot{\sigma}(F)$. $\square$

Defn F is Euclidean if F is formally real and $[F^\times : F^{\boxtimes}] = 2$.

Defn F is Pythagorean if the sum of two squares is always a square.

Prop If F is Euclidean, then F is Pythagorean with a unique ordering.

Note Converse is also true.

Pf Claim $P = F^\square$ is an ordering. Clearly have $P \subsetneq F$, $P \cdot P \subseteq F$, $P \cup (-P) = F$, so only need to show $P + P \subseteq P$, i.e. F is Pythagorean. Suffices to show $1 + y^2 \in F^{\boxtimes}$ for all $y \in F$. If $1 + y^2 \in F \setminus F^{\boxtimes} = -F^\square$, then $-1 \in \sigma(F)$ ⚡.

Uniqueness follows since $F^{\square} \subseteq \sigma(F) \subseteq P$ for all orderings. $\square$

<u>Thm</u> For all fields F, TFAE:

(1) F is Euclidean.

(2) F is formally real, but every quadratic extension of F is nonreal.

(3) $\sqrt{-1} \notin F$ and $K := F(\sqrt{-1})$ is quadratically closed (i.e. $K^{\square} = K$)

(4) $\operatorname{char}(F) \neq 2$ and $\exists$ quad extn L/F that is quadratically closed.

<u>Pf</u> $(2) \Rightarrow (1)$: For any nonsquare $a \in F$, $F(\sqrt{a})$ is nonreal, so $-a = a_1^2 + \cdots + a_n^2$ for some $a_i \in F$. Take such an eqn with n minimal (so $a_i \neq 0$, in particular). Need to show $n=1$. If $n \geq 2$, $a_1^2 + a_2^2 \notin F^{\square}$ implies $-(a_1^2+a_2^2) = b_1^2 + \cdots + b_m^2$ for some $b_j \in F$, and this contradicts formal reality of F.

$(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2)$: More work (norms, quadratic forms). $\square$

<u>Defn</u> A field F is <u>real closed</u> if F is formally real, but no proper algebraic extn of F is formally real.

<u>Cor</u> If F is real closed, then F is euclidean, has unique ordering $F^{\square}$, and $F(\sqrt{-1})$ is quadratically closed.

<u>Prop</u> Let F be a formally real field, and $\bar{F}$ its algebraic closure. Then $\exists$ real closed field R, $F \subseteq R \subseteq \bar{F}$.

<u>Pf</u> Let $\mathcal{R} = \{ L \subseteq \bar{F} \mid F \subseteq L,\ L \text{ formally real} \}$. If $\{F_\alpha\}$ is a chain in $\mathcal{R}$, then $\bigcup_\alpha F_\alpha \in \mathcal{R}$ too. By Zorn's Lemma, $\exists R \in \mathcal{R}$ that is maximal and thus real closed. $\square$

<u>Thm</u> F is formally real iff F has at least one ordering.

Pf. $\Leftarrow$: $-1 \notin P \supseteq \sigma(F)$.

$\Rightarrow$: Have an alg. extn $R \supseteq F$ that is real closed.

The unique ordering $R^{\square}$ on R induces one on F. $\square$

Fact Let $X_F = \{\text{orderings on } F\}$. Then $\bigcap_{P \in X_F} P = \sigma(F)$.

Say that the totally positive elts of F are the sums of squares.

e.g. $\mathbb{Q}(\sqrt{2})$ with $\varphi: \mathbb{Q}(\sqrt{2}) \to \mathbb{R}$, $\sqrt{2} \mapsto \sqrt{2}$, and $\varphi': \mathbb{Q}(\sqrt{2}) \to \mathbb{R}$, $\sqrt{2} \mapsto -\sqrt{2}$, induces two different orderings P, P' on $\mathbb{Q}(\sqrt{2})$. These are in fact the only two. ~~Since~~ For $\theta = 5+3\sqrt{2}$, have $\varphi(\theta), \varphi'(\theta) > 0$, so $5+3\sqrt{2} \in \sigma(\mathbb{Q}(\sqrt{2}))$.

In fact, $2(5+3\sqrt{2}) = 1^2 + (1+\sqrt{2})^2 + (1+\sqrt{2})^2 + (1+\sqrt{2})^2$.

e.g. Infinitely many orderings on $F(x)$ for F formally real.