Characterizations of real closed fields

Prop TFAE: (1) Any odd degree $f \in F[x]$ has a root in F
(2) F has no proper odd degree extns.

Pf $(2) \Rightarrow (1)$: By induction on $n = \deg(f)$. Triv for $n=1$. Assume $n > 1$. If f is irred, then $F[x]/(f)$ proper odd deg extn, ⚡. So $f = f_1 f_2$ with, say, $\deg(f_1)$ odd $< n$. But then f_1 has a root in F so f does too.

$(1) \Rightarrow (2)$: If K/F has odd deg $n > 1$, $\exists \theta \in K \setminus F$ and $\deg m_{\theta,F} = [F(\theta):F]$ is an odd integer ≥ 1. It has a root in F by (1), so ⚡. □

Fact If F is formally real, then every odd degree extn of F is as well.
(Proof via Springer's Thm on quadratic forms.)

Cor If F is real closed, then any odd deg poly $f \in F[x]$ has a root in F. □

Thm TFAE: (1) F is real closed.
(2) F is Euclidean and every odd-degree polynomial in $F[x]$ has a root in F
(3) $\sqrt{-1} \notin F$ and $K = F(\sqrt{-1})$ is algebraically closed.

Cor $\mathbb{R}$ is real closed and $\mathbb{C}$ is algebraically closed. □

Pf of Thm $(3) \Rightarrow (1)$: F Euclidean so F formally real. Since the only proper alg extn of F is K (which is non real), F is real closed.

$(1) \Rightarrow (2)$: ✓

$(2) \Rightarrow (3)$: Have K quadratically closed. If $f(x) \in K[x]$ nonconstant then $f\bar{f} \in F[x]$. If $f\bar{f}$ has a root in K, then f does, so suffices to show all $g \in F[x] - F$ have a root in K. Let E be the splitting field of $(x^2+1)g$ over F, which is a Galois extn. Since F has no odd deg extns, get that $[E:F] = 2^n$. (If not a power of 2, fixed field of H = 2-Sylow subgp of $Gal(E/F)$ is odd degree.)

$$\begin{tikzcd} E \arrow[d, dash] \arrow[dr, dash] & \\ F \arrow[r, dash] & K \end{tikzcd}$$

Since K has no ~~odd deg~~ quadratic extns (K quad closed b/c F Euclidean) get that $K = E$. Since E splits $(x^2+1)g(x)$, get that g has a root in K. $\square$

Thm [Artin-Schreier] Let C be any algebraically closed field, and $F \subsetneq C$ with $[C:F] < \infty$. Then $\mathrm{char}(F) = 0$, F is real closed, and $C = F(\sqrt{-1})$.

Pf (Assuming char $F = 0$) Claim $[C:F]$ is a power of 2.

Assume for $\Rightarrow\!\Leftarrow$ that an odd prime $p \mid [C:F]$. Since C/F is finite Galois with $|Gal(C/F)| = [C:F]$ divisible by p, know $\exists H \leq Gal(C/F)$ of order p and $[C : \underset{K}{\overset{\shortparallel}{C^H}}] = p$.

Fix $\zeta = \zeta_p \in C$. Since ζ has deg $\leq p-1$ over K, get $\zeta \in K$. Thus $C = K(x)$ where $x \in C$, $x^p = a \in K$.

Let $\langle \sigma \rangle = Gal(C/K) \cong C_p$ and take $y \in C$ s.t. $y^p = x$ (so $y^{p^2} = a$). Then $\sigma(y) = \alpha y$ for some α s.t. $\alpha^{p^2} = 1$.

If $\alpha^p = 1$, then $\sigma(x) = \sigma(y)^p = y^p = x$, $\Rightarrow\!\Leftarrow$, so α is a primitive p^2 root of unity. Thus $\sigma(\alpha) = \alpha^r$ for some r rel prime to p.

Whence $\sigma^2(y) = \alpha^{r+1} y$, $\sigma^3(y) = \alpha^{r^2+r+1} y$, etc., ultimately giving $y = \sigma^p(y) = \alpha^{r^{p-1} + \cdots + r + 1} y$.

Thus $r^{p-1} + \cdots + r + 1 \equiv 0 \pmod{p^2}$. Multiplying by r, get $r^p \equiv 1 \pmod{p^2}$. In particular, $r^p \equiv 1 \pmod{p}$, so (FLT) $r \equiv 1 \pmod{p}$, $r = 1 + kp$ for some $k \in \mathbb{Z}$. But then

$$
\begin{aligned}
r^{p-1} + \cdots + r + 1 &= \frac{r^p - 1}{r - 1} \\
&= \frac{(1+kp)^p - 1}{kp} \\
&= \frac{\binom{p}{1}kp + \binom{p}{2}(kp)^2 + \binom{p}{3}(kp)^3 + \cdots + (kp)^p}{kp} \\
&= p + \binom{p}{2}kp + \underbrace{\binom{p}{3}(kp)^2 + \cdots + (kp)^{p-1}} \\
&\equiv p \pmod{p^2}
\end{aligned}
$$

manifest for

and $\binom{p}{2}kp = p\frac{(p-1)}{2}kp = \frac{k(p-1)}{2}p^2$ is a multiple of p^2 since p odd.

This contradicts $r^p \equiv 1 \pmod{p^2}$.

Now know $[C:F] = 2^n$ for some n. Claim $n = 1$.

If $n \geq 2$, get $E \subseteq L \subseteq C$ with $[C:L] = [L:E] = 2$ (by Galois thy + ~~Sylow~~ fact that gps of order p^n have subgps of order p^k $\forall\, 0 \leq k \leq n$). Get L Euclidean since C quad closed, so $\sqrt{-1} \notin L$. Then $E(\sqrt{-1})$ is another subfield of C with $[C : E(\sqrt{-1})] = 2$, so $E(\sqrt{-1})$ Euclidean, ↯ b/c $\sqrt{-1} \in E(\sqrt{-1})$.

Therefore $[C:F] = 2$. Again, $\sqrt{-1} \notin F$, so $F(\sqrt{-1}) = C$. $\square$