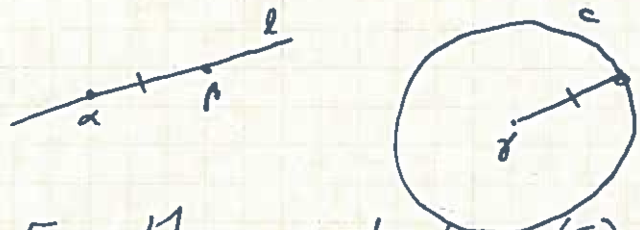
Constructible Numbers

What is a construction? Have some known points, use straightedge and compass to build lines and circles:

C1 From $\alpha \neq \beta$, can draw the line ℓ through α, β.

C2 From $\alpha \neq \beta$ and γ, draw circle C with center γ and radius the distance from α to β.



From these constructions (C) get the following points

P1 The point of intersection of distinct lines ℓ_1, ℓ_2 constructed as above

P2 The points of intersection of a line ℓ and circle C constructed as above

P3 The points of intersection of distinct circles C_1, C_2 constructed as above.

Consider the plane to be $\mathbb{C}$, start w/ #s/pts $0, 1$ to get

Defn $\alpha \in \mathbb{C}$ is <u>constructible</u> if there is a finite sequence of straightedge & compass constructions using C1, C2, P1, P2, P3 that begins w/ $0, 1$ and ends with α.

TPS Construct
- 2
- $n \in \mathbb{Z}$
- vertical axis
- $\pm i$, $\mathbb{Z}i$.

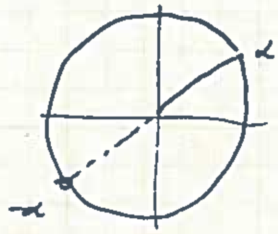
e.g. $\zeta_n = e^{2\pi i/n}$ constructible iff regular n-gon can be constructed by ruler and compass.

Thm $\mathcal{C} := \{\alpha \in \mathbb{C} \mid \alpha \text{ is constructible}\}$ is a subfield of $\mathbb{C}$. Furthermore

(a) Let $\alpha = a + ib$, $a, b \in \mathbb{R}$. Then $\alpha \in \mathcal{C}$ iff $a, b \in \mathcal{C}$.

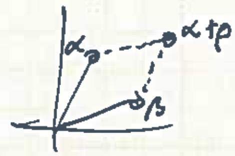
(b) $\alpha \in \mathcal{C} \Longrightarrow \sqrt{\alpha} \in \mathcal{C}$.

Pf Take $\alpha \in \mathcal{C} \setminus 0$



$\Longrightarrow -\alpha \in \mathcal{C}$

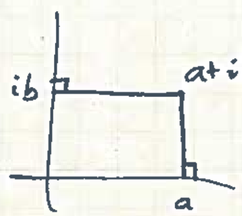
For $\alpha, \beta \in \mathcal{C}$ not collinear with 0



— intersect $|\beta|$ circle ~~thru~~ w/ cen[ter] α with $|\alpha|$ circle ~~thru~~ β. w/ center

Check Collinear case.

This proves $\mathcal{C}$ is a subgp of $\mathbb{C}$ under $+$. Now prove (a):

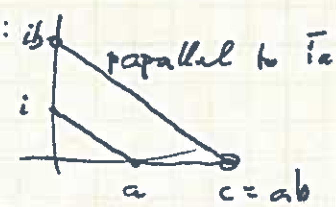


$a+ib \in \mathcal{C} \Longrightarrow a, ib \in \mathcal{C}$

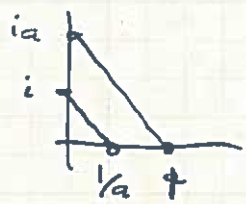
Circle w/ center ~~i~~ 0 radius $|ib| = |b|$ gives $\pm|b|$, one of these is $b \in \mathcal{C}$.

Check $a, b \in \mathcal{C} \cap \mathbb{R} \Longrightarrow a + ib \in \mathcal{C}$. So (a) ✓

Now take $a, b \in \mathcal{C} \cap \mathbb{R}_{>0}$:



$\Longrightarrow ab \in \mathcal{C}$



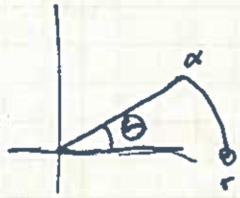
$\Longrightarrow \frac{1}{a} \in \mathcal{C} \Longrightarrow \mathcal{C} \cap \mathbb{R}$ subfield of $\mathbb{R}$

$$(a+ib)(c+id) = (ac - bd) + i(ad + bc)$$

$$\frac{1}{a+ib} = \frac{a}{a^2+b^2} + i\,\frac{-b}{a^2+b^2}$$

$\Big\} \Longrightarrow \mathcal{C}$ a field

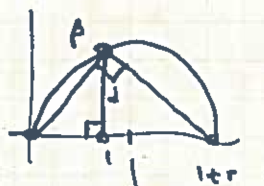
For (b), consider $\alpha = re^{i\theta}$, $r = |\alpha| > 0$, $\alpha \in C$.



angle bisection $\rightsquigarrow e^{i\frac{\theta}{2}}$ constr.

$\Rightarrow r \in C$

So just need $\sqrt{r} \in C$:



$\frac{1}{d} = \frac{d}{r} \Rightarrow d^2 = r \Rightarrow d = \sqrt{r} \in C$.

e.g. $\zeta_5 = \frac{-1+\sqrt{5}}{4} + \frac{i}{2}\sqrt{\frac{5+\sqrt{5}}{2}} \in C$ so the regular pentagon is constructible.

Thm For $\alpha \in \mathbb{C}$, $\alpha \in C$ iff $\exists$ subfields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \mathbb{C}$ with $\alpha \in F_n$ and $[F_i : F_{i-1}] = 2$ for $1 \le i \le n$.