MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE WEDNESDAY 12 DECEMBER

Complete five of the following problems based on your peers' final presentations.

Problem 1 (JR). Let $G = K_4$, the Klein four group with matrix representation

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Use Noether's theorem to determine the fixed field $K(x,y)^G$.

Problem 2 (Zichen). Let $\alpha \neq 0, 1$ be an algebraic number. Use the Lindemann-Weierstrass theorem to prove that e^α, $\log \alpha$, $\sin \alpha$, and $\cos \alpha$ are transcendental.

Problem 3 (Alex). Prove that if the n-gon is *neusis* constructible and $p = 2^u 3^v + 1$ is an Pierpont prime not equal to 3, then p^2 does not divide n. You may assume that

(a) if the n-gon is *neusis* constructible and d divides n, then the d-gon is *neusis* constructible; and
(b) for $\alpha \in \mathbb{C}$, α is *neusis*-constructible if and only if there exist subfields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \mathbb{C}$ such that $\alpha \in F_n$, and $[F_i : F_{i-1}] = 2$ or $[F_i : F_{i-1}] = 3$ for $1 \leq i \leq n$.

Problem 4 (Pallavi). A *Pieront prime* is a prime $p > 3$ of the form $p = 2^k 3^l + 1$. Prove that a regular n-gon can be constructed using origami if $n = 2^a 3^b p_1 \cdots p_s$, where $a, b \geq 0$ and $p_1 \cdots p_s$ are distinct Pierpont primes.

Problem 5 (Tristan). Let $L = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots)$ where $\{p_1, p_2 \ldots\}$ is the set of all primes. What is the Galois group of L/F?

Problem 6 (Nick). Let G be an Abelian group and M a G-module on which G acts trivially. Show that $H^1(G, M) \cong \mathrm{Hom}(G, M)$. [See the Google Drive folder of final papers for a longer text from Nick treating this problem.]

Problem 7 (Miles). Explain why we can find the Heisenberg group p_+^{1+2} as a subgroup of $\mathrm{AGL}_1(\mathbb{F}_p) \wr \mathrm{AGL}_1(\mathbb{F}_p)$ for $p > 2$ prime. (*Hint*: Look at Proposition 2.0.6 of Miles's paper, and think of p_+^{1+2} as the semidirect product $\mathbb{Z}/p\mathbb{Z}^2 \rtimes \mathbb{Z}/p\mathbb{Z}$ with action given by $x \cdot (a, b) = (a, b + ax)$.)

Problem 8 (Max).

Problem 9 (Caroline). Prove that 2 is ramified in $\mathbb{Z}[i]$.

Problem 10 (Anton). Let the vertices of an octahedron in $\mathbb{R}^3$ be $(\pm 1, \pm 1, \pm 1) \subset S^2$. By steriographic projection onto $\widehat{\mathbb{C}}$, we see that these correspond to $0, \pm 1, \pm i, \infty$. Let $r_1, r_2, r_3 \in Rot(S^2)$, where r_1 is the rotation taking $(0,0,1)$ to $(0,0,-1)$, r_2 is the rotation taking $(1,0,0)$ to $(0,1,0)$, and r_3 is the rotation taking $(0,0,1)$ to $(1,0,0)$. Find $\gamma_1, \gamma_2, \gamma_3 \in GL(2, \mathbb{C})$ such that $[\gamma_1], [\gamma_2], [\gamma_3] \in PGL(2, \mathbb{C})$ correspond to these rotations. Justify your claim.

Problem 11 (Genya). We have seen one covering space of the circle, namely $p : \mathbb{R} \to S^1$ given by $p(r) = (\cos 2\pi r, \sin 2\pi r)$, which corresponds to "wrapping" the real line around the circle. Thinking of S^1 as $\{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C}$ show that the squaring map makes S^1 into a covering space of itself.

Problem 12 (Torin). Consider the polynomial $f = x^5+15x+12 \in \mathbb{Q}[x]$. Assume that $\theta_f(y)$ has a root in $\mathbb{Q}$. Then use the discriminant of f and Proposition 5.4 (in Torin's paper) to find the Galois group of f over $\mathbb{Q}$. Finally, use Theorem 4.1 (in Torin's paper) to determine if f is solvable by radicals.

Problem 13 (Livia). Let F be a field with characteristic 0, and let $g \in F[t_1, \ldots, t_n]$ be nonzero. For each $i = 1, \ldots, n$, pick a nonnegative integer N_i such that the highest power of t_i appearing in g is at most N_i, and let

$$A = \{(a_1, \ldots, a_n) | a_i \in \mathbb{Z}, 0 \leq a_i \leq N_i\}.$$

Prove that there is $(a_1, \ldots, a_n) \in A$ such that $g(a_1, \ldots, a_n) \neq 0$.