MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 9

Problem 1. Suppose F has characteristic 0 and $L = F(\gamma)$ where $\gamma^m \in F$. Suppose further that F contains a primitive m-th root of unity, ζ. This type of extension appeared in our proof of Galois's theorem, and we have already seen that L/F is Galois.

(a) Let $\sigma \in \mathrm{Gal}(L/F)$. Show there is a unique integer $0 \leq \ell \leq m-1$ such that $\sigma(\gamma) = \zeta^\ell \gamma$.
(b) Show that $\sigma \mapsto [\ell]$ defines an injective homomorphism $\mathrm{Gal}(L/F) \to \mathbb{Z}/m\mathbb{Z}$.
(c) Conclude that $\mathrm{Gal}(L/F)$ is cyclic of order dividing m.

Problem 2. Prove that A_n is generated by 3-cycles when $n \geq 3$.

Problem 3. Let F be a subfield of $\mathbb{R}$. Let a be an element of F and let $K = F(\sqrt[n]{a})$ where $\sqrt[n]{a}$ denotes a real n-th root of a. Prove that if L is any Galois extension of F contained in K, then $[L : F] \leq 2$.