MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 8

Problem 1. Let L denote the splitting field of $x^4 - 4x^2 + 2$ over $\mathbb{Q}$, so that $L = \mathbb{Q}(\sqrt{2+\sqrt{2}})$. We have previously shown that $\mathrm{Gal}(L/\mathbb{Q}) \cong C_4$. Determine all the subgroups of $\mathrm{Gal}(L/\mathbb{Q})$ and the corresponding subextensions of $L/\mathbb{Q}$.

Problem 2. Let $\zeta = \zeta_7 = e^{2\pi i/7}$, and consider the extension $L = \mathbb{Q}(\zeta)/\mathbb{Q}$. We have previously seen that L is the splitting field of $m_{\zeta,\mathbb{Q}} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. Let $\mathbb{F}_7^\times$ be the multiplicative group of nonzero congruences classes in $\mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$.

(a) Show that $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to $\mathbb{F}_7^\times$. (Consider the action of $\mathrm{Gal}(L/\mathbb{Q})$ on $\{\zeta^i \mid i = 1, \ldots, 6\}$.)
(b) Let $H = \langle -1 \rangle \leq \mathbb{F}_7^\times$. Prove that $\mathbb{Q}(\zeta + \zeta^{-1})$ is the fixed field of the subgroup of $\mathrm{Gal}(L/\mathbb{Q})$ corresponding to H.
(c) Show that the minimal polynomial of $\zeta + \zeta^{-1}$ over $\mathbb{Q}$ is $x^3 + x^2 - 2x - 1$.
(d) Show that the splitting field of $x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$ is a Galois extension of degree 3 with Galois group isomorphic to C_3.

Problem 3. Let p be prime and consider the extension $L = \mathbb{Q}(\zeta_p, \sqrt[p]{2})/\mathbb{Q}$. We have shown that $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{AGL}_1(\mathbb{F}_p)$. The group $\mathrm{AGL}_1(\mathbb{F}_p)$ has subgroups

$$T = \{\gamma_{1,b} \mid b \in \mathbb{F}_p\} \qquad \text{and} \qquad D = \{\gamma_{a,0} \mid a \in \mathbb{F}_p^\times\},$$

where $\gamma_{a,b}(u) = au + b$, $u \in \mathbb{F}_p$. Let T' and D' be the corresponding subgroups of $\mathrm{Gal}(L/\mathbb{Q})$.

(a) Show that $L^{T'} = \mathbb{Q}(\zeta_p)$.
(b) What is the fixed field of D'? What are the conjugates of this fixed field.

Problem 4. Compute the Galois groups of the following cubic polynomials:

(a) $x^3 - 4x + 2$ over $\mathbb{Q}$.
(b) $x^3 - 4x + 2$ over $\mathbb{Q}(\sqrt{37})$.
(c) $x^3 - t$ over $\mathbb{C}(t)$, t a variable.
(d) $x^3 - t$ over $\mathbb{Q}(t)$, t a variable.

Problem 5. Consider the groups A_4 and Σ_4.

(a) Show that $\{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a normal subgroup of Σ_4.
(b) Show that A_4 and Σ_4 are solvable.

Problem 6 (Bonus). Exercise 7 in Cox §8.1.