MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 6

Problem 1. Let H and N be groups and let $\varphi : H \to \mathrm{Aut}(N)$ be a homomorphism. Define an action of H on N via $h \cdot n = \varphi(h)(n)$. Let $N \rtimes H$ denote the set $N \times H$ equipped with the operation $(n, h)(n', h') = (n(h \cdot n'), hh')$.

(a) Prove that $N \rtimes H$ is a group.
(b) Prove that the map $N \rtimes H \to H$ defined by $(n, h) \mapsto h$ is a surjective group homomorphism with kernel $N \times \{e\}$.
(c) Prove that $n \mapsto (n, e)$ defines an isomorphism $N \cong N \times \{e\}$ (where $N \times \{e\}$ inherits its group structure from $N \rtimes H$).

Problem 2. Let $p \geq 3$ be prime, and let $G = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$ be the semidirect product described in class.

(a) Show that G is nonabelian.
(b) Show that $\mathbb{F}_p \times \mathbb{F}_p^\times$ is an extension[1] of $\mathbb{F}_p$ by $\mathbb{F}_p^\times$ which is not isomorphic to G.

Problem 3. Let $L = \mathbb{Q}(\sqrt{2+\sqrt{2}})$. You have previously shown that $f = x^4 - 4x^2 + 2$ is the minimal polynomial of $\sqrt{2+\sqrt{2}}$ over $\mathbb{Q}$ and that L is the splitting field of f. Show that $\mathrm{Gal}(L/\mathbb{Q}) \cong C_4$.

Problem 4. Construct the Galois closure of $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$.

Problem 5. Suppose that L/K and K/F are field extensions and that the induced extension L/F is Galois. Show that for $\sigma \in \mathrm{Gal}(L/F)$,

$$K = \sigma K \iff \mathrm{Gal}(L/K) = \sigma \, \mathrm{Gal}(L/K) \sigma^{-1}.$$

[1] A group G is an extension of H by N if there exists a surjective homomorphism $G \to H$ with kernel isomorphic to N.