MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 5

Problem 1. For $f \in \mathbb{Z}[x]$, let f_p denote the mod p reduction of f in $\mathbb{F}_p[x]$. For $f = x^7 + x + 1$, find all primes p for which f_p is not separable, and compute $\gcd(f_p, f_p')$.

Problem 2. Let F have characteristic p and let L/F be a finite extension with $p \nmid [L : F]$. Prove that L/F is separable.

Problem 3. Let L/F be a finite extension, and let $\sigma : L \to L$ be a ring homomorphism that is the identity oon F. Prove that σ is surjective. (We already know that σ is injective, so it follows that $\sigma \in \mathrm{Gal}(L/F)$ is an automorphism of L/F.)

Problem 4. Let $p_1, \ldots, p_n$ be distinct primes. Prove that $|\mathrm{Gal}(\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})/\mathbb{Q})| \leq 2^n$. *After Wednesday's class*: Prove that the order is exactly 2^n and determine the isomorphism type of $\mathrm{Gal}(\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})/\mathbb{Q})$.

Problem 5. Let $L = \mathbb{Q}(\zeta_5, \sqrt[5]{2})$ where $\zeta_5 = e^{2\pi i/5}$. Recall that $m_{\zeta_5,\mathbb{Q}} = x^4 + x^3 + x^2 + x + 1$.
(a) Show that $[L : \mathbb{Q}] = 20$.
(b) Show that L is the splitting field of $x^5 - 2$ over $\mathbb{Q}$ and conclude that $|\mathrm{Gal}(L/\mathbb{Q})| = 20$.