MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 4

Problem 1. Determine (with proof) the degree of the following extensions:
(a) $\mathbb{Q}(i, \sqrt[4]{2})/\mathbb{Q}$, and
(b) $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$.

Problem 2. Suppose that α and β are algebraic over F with minimal polynomials f and g, respectively. Prove the *Reciprocity Theorem*: f is irreducible over $F(\beta)$ if and only if g is irreducible over $F(\alpha)$.

Problem 3. In 1873, Hermite proved that Euler's constant, e, is transcendental over $\mathbb{Q}$, and in 1882, Lindemann proved that π is transcendental over $\mathbb{Q}$. It is unknown whether $\pi + e$ and $\pi - e$ are transcendental. Prove that *at least one* of these numbers is transcendental over $\mathbb{Q}$.

Problem 4. Let F be a field and let x be transcendental over F. Show that the elements of F are the only elements of $F(x)$ that are algebraic over F.

Problem 5. Let n be a positive integer. Then the polynomial $f = x^n - 2$ is irreducible over $\mathbb{Q}$ by the Eisenstein criterion for the prime 2.
(a) Determine the splitting field L of f over $\mathbb{Q}$.
(b) Show that $[L : \mathbb{Q}] = n(n-1)$ when n is prime.

Problem 6. Determine (with proof) whether the following extensions are normal:
(a) $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ where $\zeta_n = e^{2\pi i/n}$, and
(b) $\mathbb{F}_3(t, \alpha)/\mathbb{F}_3(t)$ where t is a variable and α is a root of $x^3 - t$ in a splitting field.