MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 3

Throughout, F is a field.

Problem 1. Let $f \in F[x]$ be irreducible and suppose $g \in F[x]$ is not in the ideal (f). Show that there are polynomials $A, B \in F[x]$ such that $Af + Bg = 1$. Prove that $B + (f)$ is the multiplicative inverse of $g + (f)$ in $L = F[x]/(f)$.

Problem 2. For $f = a_0x^n + a_1x^{n-1} + \cdots + a_n \in \mathbb{C}[x]$, define $\overline{f} := \overline{a_0}x^n + \overline{a_1}x^{n-1} + \cdots + \overline{a_0}$. Prove that $\overline{fg} = \overline{f}\,\overline{g}$ for $f, g \in \mathbb{C}[x]$. Also prove that if $\alpha \in \mathbb{C}$ and $\overline{f}(\alpha) = 0$, then $f(\overline{\alpha}) = 0$.

Problem 3. Prove that the Fundamental Theorem of Algebra is equivalent to the assertion that every nonconstant polynomial in $\mathbb{R}[x]$ is a product of linear and quadratic factors with real coefficients.

Problem 4. (a) For a field extension L/F and $\alpha \in L$ show that the multiplication-by-α map $\alpha\cdot : L \to L$ is an F-linear transformation.

(b) Let $L = \mathbb{Q}(\sqrt{D})$ for some squarefree integer D. Show that L has ordered basis 1, $\sqrt{D}$ over $\mathbb{Q}$.

(c) Let $\alpha = a + b\sqrt{D}$, $a, b \in \mathbb{Q}$. Determine the matrix for $\alpha\cdot$ with respect to the ordered basis 1, $\sqrt{D}$.

(d) Prove that the map $a + b\sqrt{D} \mapsto$ the matrix for $(a + b\sqrt{D})\cdot$ with respect to 1, $\sqrt{D}$ induces a ring isomorphism beween L and its image in $M_{2\times 2}(\mathbb{Q})$.

Problem 5. (a) Prove that $x^3 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$.

(b) Show that $x^4 - 10x^2 + 1$ is not irreducible over $\mathbb{Q}(\sqrt{3})$. (Recall that we asserted, without proof, that this polynomial was the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$, and in particular *is* irreducible over $\mathbb{Q}$.)

Problem 6. Determine, with proof, the minimal polynomial of $1 + \sqrt{-2}$ over $\mathbb{Q}$.