MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 2

Throughout, F is a field.

Problem 1. Show that the polynomial

$$(x_1 + x_2 - x_3 - x_4)(x_1 + x_3 - x_2 - x_4)(x_1 + x_4 - x_2 - x_3)$$

is symmetric in $F[x_1, x_2, x_3, x_4]$ and use the algorithm presented in class to express it as a polynomial in σ_1, σ_2, σ_3, and σ_4.

Problem 2. Express $\sum_{i \neq j} x_i^2 x_j \in F[x_1, \ldots, x_n]$ in terms of elementary symmetric polynomials.

Problem 3. Read about Newton's identities on p.38 of the text. Use them to solve Exercise 18 on p.42 of the text.

Problem 4. Let $f \in F[x]$ be monic and assume that $f = (x - \alpha_1) \cdots (x - \alpha_n)$ in some field L containing F. Prove that $\Delta(f) \neq 0$ if and only the α_i are distinct. This shows that f has distinct roots if and only if its discriminant is nonzero.

Problem 5. Recall from class that $\sqrt{\Delta}$ can be expressed by the determinant of the Vandermonde matrix. Use this (and the fact that $\det(A^T) = \det(A)$ for any square matrix A) to prove that

$$\Delta = \det \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_n & p_{n+1} & \cdots & p_{2n-2} \end{pmatrix}$$

where $p_i = x_1^i + \cdots + x_n^i$ is the sum of i-th powers of the variables.

Problem 6. Suppose that $\varphi : F \to L$ is a ring homomorphism and F and L are both fields. Prove that φ is injective and induces an isomorphism $F \cong \varphi(F)$.

Problem 7 (Bonus). Implement an algorithm in `sage` which expresses symmetric polynomials in terms of the σ_i. Follow the instructions at `https://doc.sagemath.org/html/en/developer/` in order to add the function to the official distribution of `sage`.