MATH 342 – PRESENTATIONS

As outlined in the syllabus, Math 342 students will create and present 20-minute lectures on topological topics during the end of the term. Additionally, they will write and typeset detailed notes (3–8 pages) to accompany their lecture.

Over spring break, you are expected to create a topic proposal for your presentation. This should be a short (1–2 paragraphs) note typeset in LaTeX describing your topic, your interest in it, and a sketch of goals for your presentation.[1] Below is a non-exhaustive list of potential topics. Feel free to propose a topic which is not on this list.

» **Urysohn's lemma.** This lemma (really a major theorem in point-set topology) gives conditions under which a continuous map $f : X \to [a, b]$ exists which takes disjoint closed sets to a and b, respectively. See §33 of Munkres.

» **Countability and separation axioms.** Do you love exotic topological spaces with no conceivable use and extremely nuanced properties? Do you want to provide counterexamples to naïve point-set conjectures at the drop of a hat? This project is for you! See §§30-31 of Munkres.

» **Metrization theorems.** A project (potentially multiple projects) for the topology student who loves analysis. Munkres §34 and Chapter 6 describes a cottage industry devoted to providing conditions under which a topological space is metrizable. Metric spaces are important, so these theorems are too!

» **Dimension theory and embeddings of manifolds.** Learn how to define the "dimension" of a topological spaces, and prove theorems about when an m-dimensional space embeds into $\mathbb{R}^n$. See §50 of Munkres.

» **Cantor and Baire space.** Cantor space $\mathscr{C} = \{0, 1\}^{\mathbb{N}}$ and Baire space $\mathscr{N} = \mathbb{N}^{\mathbb{N}}$ have many remarkable properties. Explore the "middle-thirds construction" of $\mathscr{C}$ and prove that $\mathscr{N}$ is homeomorphic to $\mathbb{R} \smallsetminus \mathbb{Q}$. Learn how $\mathscr{C}$ is the "generic" compact space and $\mathscr{N}$ is the "generic" Polish space, or present Brouwer's theorem: a topological space is homeomorphic to Cantor space if and only if it is nonempty, perfect, compact, totally disconnected, and metrizable.

» **Topological groups.** A topological group is a space G which is also a group for which the multiplication and inverse operations are continuous. Many of your favorite groups (GL_n, $SO(n)$, $SU(2)$, ...) are examples of topological groups. Set up some of the basic theory (especially quotients) and examine some interesting examples. See pp.145–146 of Munkres and §4.3 of Armstrong, *Basic topology*.

» **Stone-Čech compactification.** A *compactification* of a space X is a compact Hausdorff space Y containing X as a subspace such that the closure of X in Y is all of Y. The *Stone-Čech compactification* βX of a space X is a particularly nice compactification which is (in a precise categorical sense) initial amongst continuous functions from X to compact spaces. Construct βX, prove that it has the stated universal property, and interpret these results in the language of adjunctions. See §38 of Munkres.

» **Stone spaces and profinite sets.** The categorically inclined student may be interested to learn that the category of *Stone spaces* (compact totally disconnected Hausdorff spaces) is

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[1] If you don't know how to use LaTeX, this is a great opportunity to get familiar with it! LaTeX learning resources are available on the course website.

equivalent to the category of pro-objects in the category of finite sets. Both of these are in turn dual to the category of Boolean algebras. These results depend on Stone-Čech compactification, so this might work better as one half of a "team" project in which one student presents compactification and the other does Stone spaces.

» **Compactly generated weakly Hausdorff spaces.** The category of topological spaces is not "cartesian closed," meaning that it does not have good function objects which are related in a particular fashion to the product. Learn what cartesian closed, complete, and cocomplete mean, learn what a compactly generated weakly Hausdorff space is, and learn something about why compactly generated weakly Hausdorff spaces have all these properties. See Steenrod, *A convenient category of topological spaces*.

» **Higher homotopy groups.** The path-connected components $\pi_0(X)$ and fundamental group $\pi_1(X)$ are extremely important invariants which we will study in class. These fit into an infinite family of invariants, the homotopy groups $\pi_n(X)$ for $n \in \mathbb{N}$. The group $\pi_n(X)$ consists of homotopy classes of continuous maps $S^n \to X$. Define and describe the group structure on $\pi_n(X)$. Then undertake some subset of the following projects:
  - Prove that $\pi_n(X)$ is abelian for $n \geq 2$.
  - Show that $\pi_k(S^n) = 0$ for $k < n$ and $\pi_n(S^n) = \mathbb{Z}$ for $n \geq 1$.
  - Define (and visualize!) the Hopf map η and prove that it generates $\pi_3(S^2) \cong \mathbb{Z}$.

Warning: Topic proposals are subject to approval by the instructor, and you may be asked to switch topics for a number of reasons, including but not limited to inappropriateness of topic and duplication of topic by multiple students.