MATH 342: TOPOLOGY
WEEK 3 HOMEWORK SUPPLEMENT

Problem 1. (*Note*: This problem counts as three problems.) Define a *closure operator* on a set X to be a function $\overline{(\ )} : 2^X \to 2^X$ such that for all $A, B \in 2^X$,

» $\overline{\varnothing} = \varnothing$ (closure is *nullary*),
» $A \subseteq \overline{A}$ (closure is *extensive*),
» $\overline{A \cup B} = \overline{A} \cup \overline{B}$ (closure *preserves binary unions*), and
» $\overline{\overline{A}} = \overline{A}$ (closure is *idempotent*).

Prove the following statements:

(a) Closure is *extensive*, *i.e.*, if $A \subseteq B \subseteq X$, then $\overline{A} \subseteq \overline{B}$.
(b) If X is a topological space, then
$$\overline{A} := \bigcap_{C \supseteq A \text{ closed}} C$$
is a closure operator.
(c) Given a closure operator $\overline{(\ )}$ on a set X, define $C \subseteq X$ to be *closed* if and only if $\overline{C} = C$. Then define $U \subseteq X$ to be *open* if and only if $X \smallsetminus U$ is closed. Prove that the collection of open subsets of X forms a topology on X.
(d) If a closure operator $\overline{(\ )}$ induces a topology τ which in turn induces a closure operator $\overline{(\ )}_\tau$, then $\overline{(\ )} = \overline{(\ )}_\tau$.
(e) If a topology τ induces a closure operator $\overline{(\ )}$ which in turn induces a topology $\overline{\tau}$, then $\tau = \overline{\tau}$.
(f) Define continuity in terms of closure operators and then prove that your definition matches the definition in terms of open sets when you look at the topologies induced by the closure operators.

Problem 2. Given a group G, define a category $\bullet_G$ with $\mathrm{Ob}\,\bullet_G = \{*\}$ a singleton set and with morphism set G. (Every morphism has source and target $*$.) Composition in $\bullet_G$ is given by multiplication in G (*i.e.*, for $g, h \in G$, $g \circ h = g \cdot h$ where $\cdot$ is the product in G).

(a) Prove that $\bullet_G$ is a category.
(b) Determine which morphisms in $\bullet_G$ are isomorphisms.

Problem 3. Given a category $\mathscr{C}$ and $x \in \mathrm{Ob}\,\mathscr{C}$, let $\mathrm{Aut}(x) \subseteq \mathscr{C}(x, x)$ denote the isomorphisms in $\mathscr{C}$ with source and target both x; this is called the *automorphism group* of x.

(a) Prove that composition in $\mathscr{C}$ restricted to $\mathrm{Aut}(x)$ does indeed make $\mathrm{Aut}(x)$ a group.
(b) For G a group and $*$ the unique object in $\bullet_G$, determine $\mathrm{Aut}(*)$.
(c) For a field k, recall the category Mat_k from the notes. For $n \in \mathrm{Ob}\,\mathrm{Mat}_k$, find a more familiar name or description of $\mathrm{Aut}(n)$. Are there any isomorphisms in $\mathrm{Mat}_k(m, n)$ for $m \neq n$?