MATH 342: TOPOLOGY
WEEK 2 HOMEWORK SUPPLEMENT

Problem 1. For sets A and B and a function $f : A \to B$ we defined functions

$$f_* : 2^A \to 2^B \qquad \text{and} \qquad f^* : 2^B \to 2^A$$

given by

$$f_*(U) = \{f(u) \mid u \in U\} \qquad \text{and} \qquad f^*(V) = \{a \in A \mid f(a) \in V\}.$$

for $U \subseteq A$ and $V \subseteq B$.

(a) For sets B and C and a function $g : B \to C$, prove that

$$(g \circ f)_* = g_* \circ f_* \qquad \text{and} \qquad (g \circ f)^* = f^* \circ g^*.$$

(b) Let $1_A : A \to A$ denote the identity function on A. Describe $(1_A)_*$ and $(1_A)^*$.

Problem 2. Given a function $f : A \to B$, how are injectivity and surjectivity of f related to injectivity and surjectivity properties of f_* and f^*? (A complete solution will determine all the implications which are true and then prove them.)

Problem 3. Given a function $f : A \to B$ and $U \subseteq A$, $V \subseteq B$, prove that $f_*(U) \subseteq V$ if and only if $U \subseteq f^*(V)$. (Later we will use Problems 1 and 3 to describe $(\ )_*$ and $(\ )^*$ as *adjoint functors*.)

Problem 4. Recall that in class we defined a topology on the set $\mathbb{Z}$ of integers where the open sets were precisely the $U \subseteq \mathbb{Z}$ such that $U = \varnothing$ or for every $b \in U$ there exists a positive integer a such that $a\mathbb{Z} + b \subseteq U$. Note the following two facts:

(i) Any nonempty open set is infinite.

(ii) Any set $a\mathbb{Z} + b$ (where $a, b \in \mathbb{Z}$, $a > 0$) is closed (in addition to being open).

Fact (i) is obvious from the definition of open sets in $\mathbb{Z}$.

(a) Prove that fact (ii) is true by writing $a\mathbb{Z} + b$ as the complement of an open set.

(b) Let $\mathbb{P}$ denote the set of positive prime integers and note that

$$\mathbb{Z} \smallsetminus \{\pm 1\} = \bigcup_{p \in \mathbb{P}} p\mathbb{Z} + 0$$

since every integer other than ± 1 is divisible by a prime. Use facts (i) and (ii) along with properties of topologies to conclude that $\mathbb{P}$ is an infinite set.

You just used topology to prove that there are infinitely many prime numbers!