MATH 342: TOPOLOGY
EXAM 1 REVIEW QUESTIONS

Our first exam is in-class on Friday. You will have 50 minutes to answer four questions. **One of the exam questions will be drawn from these five review questions.** The exam is closed book, closed notes, but you may bring one two-sided 8.5"×11" 'cheat sheet' to the exam containing whatever information you choose. Dave Perkinson will review these problems and answer questions during Wednesday's class. If you have any additional questions, please contact me via email at `ormsbyk@reed.edu` and I will respond promptly.

Question 1. Let X be a topological space. For a subset $A \subseteq X$, define the *interior* of A to be

$$A^\circ := \bigcup_{U \subseteq A \text{ open}} U$$

where the union is taken over all open subsets of A.

(a) Prove that A° is open and that it is the largest open subset of A (meaning that if V is open in X and a subset of A, then $V \subseteq A^\circ$).
(b) Prove that $A^\circ \cap B^\circ = (A \cap B)^\circ$.
(c) Prove that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.
(d) Show that, in general, equality does not hold in (b) by constructing A, B, and X such that $A^\circ \cup B^\circ \subsetneq (A \cup B)^\circ$.

Question 2. (a) Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = (x, y_0) \qquad \text{and} \qquad g(y) = (x_0, y)$$

are embeddings. (*Hint*: The easiest proof uses that $X \times Y$ is a product in the category Top.)
(b) Prove that the projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are open maps, *i.e.*, p_X and p_Y are continuous functions that take open sets to open sets under direct image.

Question 3. Let X and X' denote the same set endowed with topologies τ and τ', respectively. Let $1 : X' \to X$ be the identity function.

(a) Show that 1 is continuous if and only if τ' is finer than τ.
(b) Show that 1 is a homeomorphism if and only if $\tau' = \tau$.

Question 4. Let A be a subset of Y which is a subset of X and give Y the subspace topology. Prove that the subspace topology which A inherits from X is the same as the subspace topology it inherits from Y.

Question 5. (a) Prove that $\mathscr{B} = \{(a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ is a basis generating the standard topology on $\mathbb{R}$.
(b) Prove that $\mathscr{B}$ is countable. Use this fact to show that the standard topology on $\mathbb{R}^n$ has a countable basis.
(c) Prove that $\mathscr{C} = \{[a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ is a basis generating a topology on $\mathbb{R}$ which is different from the lower limit topology $\mathbb{R}_\ell$.