# TOPOLOGY FROM THE CATEGORICAL VIEWPOINT

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One of the primary insights of twentieth century mathematics is that *objects* should not be studied in isolation. Rather, to understand objects we must also understand *relationships* between objects. Topologies offer one notion of 'relation', organizing the points of a set into neighborhoods. As an abstraction of geometry, topology is immensely successful, but it fails to capture a second, more structural notion of relation.

*Category theory* organizes objects by their transformations. Once these notions of object and relation are abstracted, it becomes possible to compare and contrast different fields of mathematics. Theorems about wildly different mathematical objects are often identical in their categorical content. In this manner, category theory becomes a meta-mathematical tool for both identifying and conjecturing structural results.

These notes aim to introduce category theory in parallel with James Munkres's *Topology*. We will use this language to motivate definitions and interpret theorems. Moving into the algebro-topological portion of the course, this language will become even more important as we use the fundamental group *functor* to compare the categories of topological spaces and groups.

## 1. Categories

We begin with a motivating example. Sets are a type of mathematical object. Sets are related by functions. Each function has a domain (source) and codomain (target). In standard notation, $f : A \to B$ denotes a function with domain $A$ and codomain $B$. Of course, functions admit composition: given $g : B \to C$ and $f : A \to B$ we can form $g \circ f : A \to C$ by assigning $g(f(a))$ to $a \in A$. This composition is associative: if $f : A \to B$, $g : B \to C$, and $h : C \to D$ are functions, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Moreover, each set (even the empty set!) admits an identity function $1_A : A \to A$. This function takes each $a \in A$ to $a$ and satisfies the following property: for each $g : A \to B$ and each $f : C \to A$,

$$g \circ 1_A = g \quad \text{and} \quad 1_A \circ f = f.$$

In the following definition, we will see that categories consist of objects, morphisms (with source and target objects), composition, and identity morphisms satisfying associativity and identity properties. As such, sets and functions form our first example of a category.

**Definition 1.1.** A *category* $\mathscr{C}$ consists of a collection of *objects* $\mathrm{Ob}\,\mathscr{C}$ and a collection of *morphisms* $\mathrm{Mor}\,\mathscr{C}$ along with assignments $s, t : \mathrm{Mor}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{C}$ (called the *source* and *target* maps). Let $\mathscr{C}(x, y) \subseteq \mathrm{Mor}\,\mathscr{C}$ denote the collection of morphisms with source $x \in \mathrm{Ob}\,\mathscr{C}$ and target $y \in \mathrm{Ob}\,\mathscr{C}$. Then for each $x, y, z \in \mathrm{Ob}\,\mathscr{C}$, $\mathscr{C}$ is also equipped with a *composition*

$$\begin{aligned} \circ : \mathscr{C}(y,z) \times \mathscr{C}(x,y) &\to \mathscr{C}(x,z) \\ (g, f) &\mapsto g \circ f. \end{aligned}$$

Additionally, for each $x \in \mathrm{Ob}\,\mathscr{C}$ there is an *identity morphism* $1_x \in \mathscr{C}(x, x)$. This data must satisfy the following properties:

(associativity) For objects $x, y, z, w$ and morphisms $f \in \mathscr{C}(x, y)$, $g \in \mathscr{C}(y, z)$, and $h \in \mathscr{C}(z, w)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(identity) For each $x, y \in \operatorname{Ob}\mathscr{C}$, $f \in \mathscr{C}(x,y)$, and $g \in \mathscr{C}(y,x)$, we have

$$f \circ 1_x = f \qquad \text{and} \qquad 1_x \circ g = g.$$

It is useful to think about categories diagrammatically. These diagrams use letters to represent objects, and labelled arrows to represent morphisms. So if $x, y \in \operatorname{Ob}\mathscr{C}$ and $f \in \mathscr{C}(x,y)$, we may draw

$$x \xrightarrow{f} y$$

in order to represent that $f$ is a morphism in with source $x$ and target $y$. (Note that we have dropped $\mathscr{C}$ from our notation here: usually it will be clear from context which category we are working in.) We may also write $f : x \to y$ to represent $f \in \mathscr{C}(x,y)$.

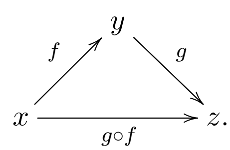
Now suppose that in addition to $f$ we also have

$$y \xrightarrow{g} z,$$

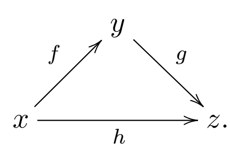
a morphism with source $y$ and target $z$. Composition tells us that we then get a new morphism

$$x \xrightarrow{g \circ f} z.$$

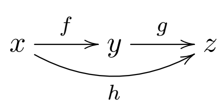
We can put all of this information into a single *commutative diagram*




When we say that a diagram




*commutes*, we are saying precisely that $h = g \circ f$. The geometric presentation of the diagram is unimportant, and we could just as easily draw commutative diagrams

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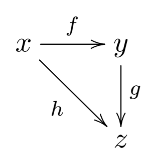
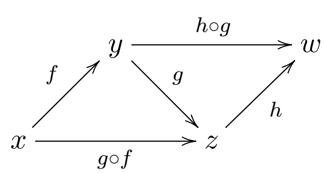
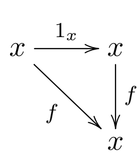



to communicate that $h = g \circ f$.

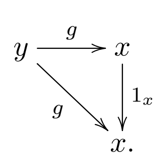
We can use diagrams to express the axioms for a category. Associativity becomes




so we can interpret this axiom as saying that we can paste together commutative triangles to produce commutative quadrilaterals. The identity axiom becomes




and



Without realizing it, you have been working with categories for a long time. Consider the following examples.

**Example 1.2.** We have already mentioned that sets and functions form a category. We denote this category $\mathrm{Set}$. Similarly, there is a category $\mathrm{FinSet}$ with objects finite sets and morphisms functions between finite sets.

**Example 1.3.** Let $k$ be a field and let $\mathrm{Vect}_k$ have objects $k$-vector spaces and morphisms $k$-linear transformations. Since linear transformations compose (in the set-theoretic sense) to give new linear transformations, $\mathrm{Vect}_k$ is also a category. (It is obvious that the identity function $1_V : V \to V$ is linear.) We can also consider the category $\mathrm{FinVect}_k$ of finite-dimensional $k$-vector spaces and $k$-linear transformations.

**Example 1.4.** The *empty category* $\varnothing$ has no objects (*i.e.*, $\mathrm{Ob}\,\varnothing = \varnothing$) and no morphisms ($\mathrm{Mor}\,\varnothing = \varnothing$). The source, target, and composition functions are all the empty function $\varnothing \to \varnothing$ and all properties are satisfied vacuously!

**Example 1.5.** The *trivial category* $\bullet$ has a singleton set $\{*\}$ for its objects and a single morphism (necessarily the identity on $*$), $1 : * \to *$. The only composition to define is $1 \circ 1 = 1$ and this is enough to check the associativity and identity properties as well.

**Example 1.6.** Not every category has special classes of functions as morphisms. Consider $\mathrm{Mat}_k$ whose objects are the natural number $\mathbb{N} = \{0, 1, 2, \ldots\}$ and whose morphisms $\mathrm{Mat}(m, n)$ are $n \times m$ matrices with entries in a field $k$. Composition is given by matrix multiplication (check compatibility!) and the $n \times n$ identity matrix is $1_n$. Since matrix multiplication is associative, this forms a category.

If you are suspicious that this category is eerily similar to $\mathrm{FinVect}_k$, worry not: it is! After studying equivalences of categories and skeleta, you will understand exactly how similar.

The following two examples should seem completely obvious if you have already taken a course in abstract algebra. If you have not, there is no harm in skipping them.

**Example 1.7.** There are categories $\mathrm{Gp}$, $\mathrm{FinGp}$, and $\mathrm{AbGp}$ of groups, finite groups, and abelian groups, respectively. In each case, morphisms are group homomorphisms.

**Example 1.8.** There are categories $\mathrm{Ring}$ and $\mathrm{CommRing}$ of rings and commutative rings, respectively. In both cases, morphisms are ring homomorphisms. There is also a category $\mathrm{Field}$ of fields and field homomorphisms.

Finally, we come to the category of primary interest in this course, the category of topological spaces and continuous functions.

**Example 1.9.** The category $\mathrm{Top}$ has topological spaces as its objects and continuous functions as its morphisms. In order to check this, we must see that the composition of continuous functions is continuous and that identity functions are continuous. The former condition is the content of Theorem 18.2(c) in Munkres. If $X$ is a topological space and $1_X : X \to X$ is the identity function, then for any $U \subseteq X$ open, $1_X^{-1}(U) = U$ is open in $X$, so $1_X$ is continuous. The associativity and identity axioms hold because they hold for functions.

*Remark* 1.10. Of course, there is also a category whose objects are topological spaces and morphisms are arbitrary functions between underlying sets. And while we are free to define such a category, it is not of particular interest. A category is a tool for studying relationships between objects. If we choose the wrong set of relationships (*i.e.*, the wrong morphisms), then we end up with an uninteresting — or, worse yet, misleading — category.

## 2. Isomorphisms

The morphisms in a category give us a way to relate or compare objects. When are two objects "the same" in light of these comparisons? Of course, this notion of sameness is different from equality. We would like to know when two objects are "indistinguishable via morphisms" rather than when they are literally equal. The following definition provides this notion.

**Definition 2.1.** A morphism $f \in \mathscr{C}(x,y)$ is an *isomorphism* if there exists a morphisms $g \in \mathscr{C}(y,x)$ such that $g \circ f = 1_x$ and $f \circ g = 1_y$. We call $g$ a (*two-sided*) *inverse* to $f$. Objects $x, y \in \operatorname{Ob}\mathscr{C}$ are *isomorphic* if there exists an isomorphism $f \in \mathscr{C}(x,y)$. In this case we write $x \cong y$, and if $f \in \mathscr{C}(x,y)$ is an isomorphism we write $f : x \cong y$.

**Proposition 2.2.** *Isomorphism is an equivalence relation on* $\operatorname{Ob}\mathscr{C}$.

*Proof.* We first check that $\cong$ is reflexive. Indeed, $1_x : x \cong x$ for all $x \in \operatorname{Ob}\mathscr{C}$ because $1_x \circ 1_x = 1_x$ by the identity axiom.

We now check that $\cong$ is symmetric. Suppose $f : x \cong y$. Then there exists $g \in \mathscr{C}(y,x)$ such that $g \circ f = 1_x$ and $f \circ g = 1_y$. This tells us that $g : y \cong x$ as well.

Finally, we check that $\cong$ is transitive. Suppose $f : x \cong y$ and $g : y \cong z$. There exist $f' \in \mathscr{C}(y,x)$ and $g' \in \mathscr{C}(z,y)$ such that $f' \circ f = 1_x$, $f \circ f' = 1_y$, $g' \circ g = 1_y$, and $g \circ g' = 1_z$. Observe that

$$(f' \circ g') \circ (g \circ f) = (f' \circ (g' \circ g)) \circ f = (f' \circ 1_y) \circ f = f' \circ f = 1_x$$

and

$$(g \circ f) \circ (f' \circ g') = (g \circ (f \circ f')) \circ g' = (g \circ 1_y) \circ g' = g \circ g' = 1_z.$$

Thus $g \circ f : x \cong z$, as desired. □

In the following sequence of propositions, we identify which morphisms are isomorphisms in some of the categories we introduced in §1.

**Proposition 2.3.** *A function $f \in \operatorname{Set}(A,B)$ is an isomorphism if and only if it is a bijection.*

*Proof.* First suppose that $f$ is an isomorphism, in which case there exists $g \in \operatorname{Set}(B,A)$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Suppose that $f(a) = f(a')$. Then, applying $g$ to both sides, we get that $a = a'$, so $f$ is injective. Given $b \in B$ we see that $f(g(b)) = b$, so $f$ is surjective as well, and hence a bijection.

Now suppose that $f$ is a bijection. Given $b \in B$, surjectivity of $f$ implies that there exists an $a \in A$ such that $f(a) = b$. By infectivity of $f$, this $a$ is unique. We may thus define $g : B \to A$ by the rule $g(b) = a$ (for $a$ the unique element of $A$ such that $f(a) = b$). It is simple to check that $g \circ f = 1_A$ and $f \circ g = 1_B$, so $f : A \cong B$. □

**Proposition 2.4.** *A linear transformation $L \in \operatorname{Vect}_k(V,W)$ is an isomorphism if and only if it is bijective as a function.*

*Proof.* First suppose that our linear transformation $L : V \to W$ is a bijective function. As we saw in the previous proof, there exists a function (but is it a linear transformation?) $M : W \to V$ such that $M \circ L = 1_V$ and $L \circ M = 1_W$. For $w, w' \in W$ suppose that $M(w) = v$ and $M(w') = v'$. Then $L(v) = w$ and $L(v') = w'$, so for a scalar $\lambda \in k$ we have $L(v + \lambda v') = w + \lambda w'$ by linearity of $L$. By the definition of $M$ it follows that

$$M(w + \lambda w') = v + \lambda v' = M(w) + \lambda M(w').$$

We conclude that $M \in \operatorname{Vect}_k(W,V)$ (*i.e.*, that $M$ is $k$-linear) so $L : V \cong W$.

Now suppose that $L \in \operatorname{Vect}_k(V,W)$ is an isomorphism. Then, since linear transformations are just special sorts of functions, we also have that $L \in \operatorname{Set}(V,W)$ is an isomorphism. The previous proposition then implies $L$ is a bijection. □

It is tempting to now guess that isomorphisms in $\mathrm{Top}$ are continuous bijective functions. This is false! Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle with basic opens the "open arcs" $\{e^{i\theta} \mid \theta \in (a, b), a < b\}$. Let $[0, 2\pi)$ denote the half-open interval of nonnegative real numbers less than $2\pi$ with basic opens $[0, 2\pi) \cap (a, b)$. The reader may check that the function

$$w : [0, 2\pi) \to S^1$$
$$\theta \mapsto e^{i\theta}$$

is a continuous bijection.

Do we still hope that continuous bijections are isomorphisms of topological spaces? Should an interval and a circle really be "the same"? Thankfully, $w$ is not an isomorphism. In the next paragraph we will check that its inverse, which takes the argument of a unit complex number, is not continuous. Since inverse functions are unique, it follow that there are no maps in $\mathrm{Top}(S^1, [0, 2\pi))$ which are inverses to $w$.

Let $\arg$ denote the inverse to $w$. The set $U = [0, \pi)$ is open in $[0, 2\pi)$, and $\arg^{-1}(U) = w(U) = \{e^{i\theta} \mid \theta \in [0, \pi)\}$. It is readily checked that this set is not open, whence $\arg$ is not continuous.

Oftentimes, isomorphisms in a particular category get a special name; this is the case in the category $\mathrm{Top}$.

**Definition 2.5.** An isomorphism in $\mathrm{Top}$ (*i.e.*, a continuous function $f : X \to Y$ for which there exists a continuous function $g : X \to Y$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$) is called a *homeomorphism*. If two spaces $X, Y$ are isomorphic in $\mathrm{Top}$, then we call them *homeomorphic* and write $X \cong Y$.

**Definition 2.6.** A continuous function $f \in \mathrm{Top}(X, Y)$ is called *open* if $f(U) \subseteq Y$ is open for all open $U \subseteq X$.

**Proposition 2.7.** *A map $f \in \mathrm{Top}(X, Y)$ is a homeomorphism if and only if it is an open bijection.*

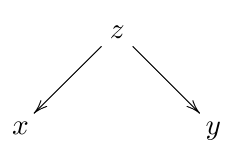
*Proof.* Suppose $f$ is an open bijection. To prove that $f$ is a homeomorphism, it suffices to show that its inverse function $g$ is continuous. For $U \subseteq X$ open, $g^{-1}(U) = f(U)$ is open in $Y$ by hypothesis, so $g$ is continuous.

Now suppose $f$ is a homeomorphism and let $g \in \mathrm{Top}(Y, X)$ be its inverse. By Proposition 2.3, $f$ is a bijection. For any open $U \subseteq X$, $f(U) = g^{-1}(U)$ is open in $Y$ because $g$ is continuous. We conclude that $f$ is open as well. ☐

In practice, we will only sometimes use Proposition 2.7 to check that a given map is a homeomorphism. Frequently, it is just as easy or easier to directly construct a continuous inverse.

## 3. Binary products

Given objects $x, y$ in a category $\mathscr{C}$ we can consider the collection of diagrams



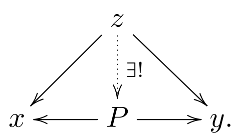

in $\mathscr{C}$. In other words, we are considering objects $z$ which "map to" both $x$ and $y$ (along with the "mappings"). A typical sort of categorical question is the following:

*Is there a universal $z$ mapping to $x$ and $y$?*

Such a universal object is referred to as the *product* of $x$ and $y$ in $\mathscr{C}$. We make this notion precise in the following definition.

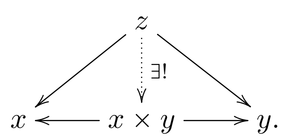
**Definition 3.1.** Suppose $P \in \operatorname{Ob}\mathscr{C}$ is equipped with morphisms $p_x \in \mathscr{C}(P, x)$ and $p_y \in \mathscr{C}(P, y)$ such that for any $f \in \mathscr{C}(z, x)$ and $g \in \mathscr{C}(z, y)$, there is a unique morphism $h \in \mathscr{C}(z, P)$ such that $p_x \circ h = f$ and $p_y \circ h = g$. Then the data $(P, p_x, p_y)$ is called the *product* of $x$ and $y$ in $\mathscr{C}$ and is denoted $x \times y$.

We can succinctly summarize the product axioms in the following commutative diagram:




Here the dotted arrow labelled "∃!" indicates that there exists (∃) a unique (!) morphism which makes the diagram commute.

Frequently, we will abuse notation and write $P = x \times y$, leaving the morphisms $p_x$ and $p_y$ implicit. In this case, the diagram reads

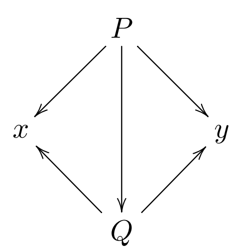



Our definition raises two obvious questions:

(1) Does $x \times y$ exist?
(2) If it does exist, is $x \times y$ unique?

The answer to (1) is no in general. There are many categories which do not have products. Nonetheless, many categories of interest do have products, including $\mathrm{Set}$ and $\mathrm{Top}$. We will study these examples in a moment.

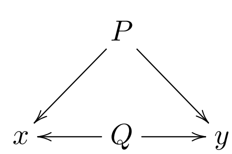
As for (2), existence of a product does imply a very good sort of uniqueness. Namely, when it exists, $x \times y$ is *unique up to unique isomorphism*. By this, we mean that if there exist $x \leftarrow P \to y$ and $x \leftarrow Q \to y$ in $\mathscr{C}$ both satisfying Definition 5.1, then there is a unique morphism $P \to Q$ such that the diagram



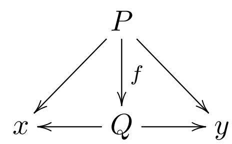

commutes, and this morphism is an isomorphism. In the categorical context, this is the best sort of uniqueness we can hope for.

**Proposition 3.2.** *If $x \times y$ exists in $\mathscr{C}$, then it is unique up to unique isomorphism in the above sense.*

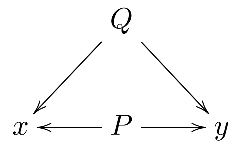
*Proof.* Suppose that $x \leftarrow P \to y$ and $x \leftarrow Q \to y$ in $\mathscr{C}$ both satisfy Definition 5.1. Then the diagram



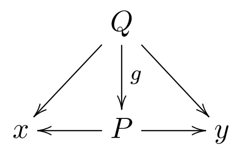
and Definition 5.1 tell us that there is a unique morphism $f : P \to Q$ such that



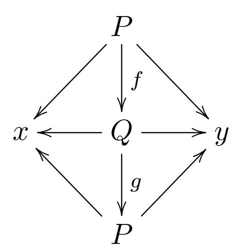

commutes. Our proof will be complete if we can show that $f$ is an isomorphism. To this end, consider the diagram



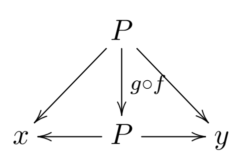

and use the definition of product to produce a unique morphism $g : Q \to P$ such that




commutes. Pasting our diagrams together, we get the commutative diagram




which tells us that

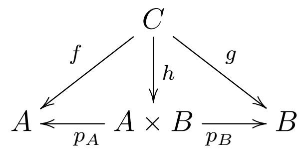



commutes. But observe that $1_P : P \to P$ in place of $g \circ f$ also makes this diagram commute. Since such a morphism is unique by definition, $g \circ f = 1_P$. The reader should work out a similar argument to show that $f \circ g = 1_Q$. Hence $f : P \cong Q$, and we have thus proven that $x \times y$ is unique up to unique isomorphism. □

We now investigate several categories that have binary products. We begin with Set, the category of sets and functions.

**Proposition 3.3.** *For sets $A$ and $B$, the cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$ along with the projection maps $p_A : A \times B \to A,\ (a, b) \mapsto a$ and $p_B : A \times B \to B,\ (a, b) \mapsto b$ is the categorical product of $A$ and $B$.*

*Proof.* Given a set $C$ and functions $A \xleftarrow{f} C \xrightarrow{g} B$, define $h : C \to A \times B$ taking $c \mapsto h(c) = (f(c), g(c))$. It is easy to check that




commutes and moreover that $h$ is the only function making the diagram commute. We conclude that $A \times B$ is indeed the categorical product of $A$ and $B$. □

**Proposition 3.4.** *If $k$ is a field and $\mathrm{Mat}_k$ is the category of natural numbers and $k$-matrices, then the categorical product of $n$ and $m$ is the natural number $n + m$ along with the morphisms*

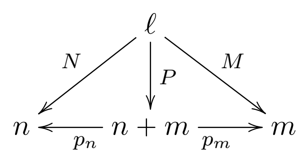
$$p_n = \begin{pmatrix} I_n & 0_{n\times m} \end{pmatrix} \qquad \text{and} \qquad p_m = \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix}$$

*where $I_j$ is the $j \times j$ identity matrix and $0_{j\times \ell}$ is the $j \times \ell$ matrix of $0$'s.*

*Proof.* Given a natural number $\ell$, an $n \times \ell$ matrix $N$, and an $m \times \ell$ matrix $M$, define $P$ to be the $(n + m) \times \ell$ matrix

$$P = \begin{pmatrix} N \\ M \end{pmatrix}.$$

This is precisely the data of a diagram




in the category $\mathrm{Mat}_k$. Observe that

$$p_n \circ P = \begin{pmatrix} I_n & 0_{n\times m} \end{pmatrix} \begin{pmatrix} N \\ M \end{pmatrix} = N \qquad \text{and} \qquad p_m \circ P = \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix} \begin{pmatrix} N \\ M \end{pmatrix} = M$$

so the diagram commutes. We leave it to the reader to check that $P$ is the unique matrix having this property. □

We now turn to the study of products in $\mathrm{Top}$, the category of topological spaces and continuous functions. We certainly suspect that the product of spaces $X$ and $Y$ has the cartesian product $X \times Y$ as its underlying set. But is there a topology on $X \times Y$ which will make it satisfy Definition 5.1? To approach this question, observe that we will need both of the projection maps

$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$$

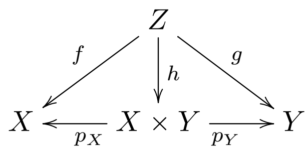
to be continuous. For $U \subseteq X$, we have $p_X^{-1}U = U \times Y$, and for $V \subseteq Y$, we have $p_Y^{-1}V = X \times V$. Thus for $U \subseteq X$ open and $V \subseteq Y$ open, we must have $U \times Y$ and $X \times V$ open in $X \times Y$. Moreover, open sets are closed under finite intersection, so

$$U \times V = (U \times Y) \cap (X \times V)$$

must be open in $X \times Y$. Since Definition 5.1 does not seem to put additional restrictions on the topology, our best guess is that the topology on $X \times Y$ is generated by the sets $U \times V$. (The reader should check that $\{U \times V \mid U \subseteq X \text{ and } V \subseteq Y \text{ open}\}$ is a basis.)

**Theorem 3.5.** *For topological spaces $X$ and $Y$, let $\tau_{X\times Y}$ be the topology on the cartesian product $X \times Y$ generated by the basis of sets $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open. Then $(X, \tau_{X\times Y})$ equipped with the standard projection maps $p_X$ and $p_Y$ is the categorical product of $X$ and $Y$.*

*Proof.* First note that $p_X$ and $p_Y$ are continuous by the discussion preceding the statement of the theorem. Given a space $Z$ and continuous functions $X \xleftarrow{f} Z \xrightarrow{g} Y$, let $h : Z \to X \times Y$ be the function taking $z \mapsto (f(z), g(z))$. By Proposition 5.5, $h$ is the unique function making the diagram




commute (as a diagram of sets and functions). If we prove $h$ is continuous, then we will be done with our proof.

Since $\tau_{X\times Y}$ is generated by the basis of sets $U \times V$ ($U \subseteq X$ and $V \subseteq Y$ open), it suffices to check that $h^{-1}U \times V \subseteq Z$ is open. By an easy computation,

$$h^{-1}U \times V = f^{-1}U \cap g^{-1}V.$$

Since $f$ and $g$ are continuous and finite intersections of open sets are open, we see that $h^{-1}U \times V$ is open, completing our proof. ☐

We conclude this section by mentioning that there are many categories which do not have binary products. For instance, suppose $\mathscr{C}$ is a category in which $\mathscr{C}(x, y) = \varnothing$ whenever $x \neq y$. An example of such a category is $\Sigma_\infty$ which has the sets $\underline{n} = \{0, 1, \dots, n-1\}$ as objects and $\Sigma_\infty(\underline{n}, \underline{m})$ equal to the set of bijections $\underline{n} \to \underline{m}$. In order for $x \times y$ to exist in $\mathscr{C}$, we must have morphisms $x \leftarrow x \times y \to y$, and in a category such as $\Sigma_\infty$, we won't have any such morphisms when $x \neq y$!

## 4. Monomorphisms and subobjects

In §2 we studied isomorphisms, the morphisms in a category which identify when objects are categorically indistinguishable. In this section, we turn to *monomorphisms*, another special class of morphisms.

**Definition 4.1.** A morphism $i : x \to y$ in a category $\mathscr{C}$ is a *monomorphism* if for all morphisms $f_1, f_2 : z \to x$ in $\mathscr{C}$, the equality $i \circ f_1 = i \circ f_2$ implies that $f_1 = f_2$. (In other words, $i$ is *left-cancellative* with respect to composition in $\mathscr{C}$.) When $i$ is a monomorphism we say that it is *monic* and write $i : x \hookrightarrow y$.

We can think of this definition as saying that whatever the composites $i \circ f_1$ and $i \circ f_2$ can "see" of $y$, in fact happened in $x$ before postcomposition with $i$. A few examples will help us hone our intuition about monomorphisms.

**Proposition 4.2.** *The monomorphisms in* Set *are precisely the injective functions.*

*Proof.* Suppose that $i : A \hookrightarrow B$ is a monomorphism. For elements $a_1$ and $a_2$ of $A$, consider the functions $f_1, f_2 : \{*\} \to A$ taking $* \mapsto a_1$ and $* \mapsto a_2$, respectively. Suppose that $i(a_1) = i(a_2)$. We may reinterpret this condition as saying that $i \circ f_1 = i \circ f_2$ since the first composite takes $* \mapsto i(a_1)$ while the second takes $* \mapsto i(a_2)$. Since $i$ is monic, we have $f_1 = f_2$, which is equivalent to $a_1 = a_2$.

Now suppose that $i : A \to B$ is injective and that there are functions $f_1, f_2 : C \to A$ such that $i \circ f_1 = i \circ f_2$. Then for each $c \in C$, $i(f_1(c)) = i(f_2(c))$. Since $i$ is injective, we learn that $f_1(c) = f_2(c)$, whence $f_1 = f_2$, so $i$ is monic. ☐

**Proposition 4.3.** *For $k$ a field, the monomorphisms in* $\mathrm{Vect}_k$ *are the injective $k$-linear maps.*

*Proof.* Suppose that $i : V \hookrightarrow W$ is a monomorphism in $\mathrm{Vect}_k$. Given a nonzero element $v \in V$, let $f_v : k \to V$ be the $k$ linear map taking $1 \mapsto v$ and let $0 : k \to V$ be the map taking everything in

$k$ to $0 \in V$. Suppose for contradiction that $i(v) = 0$. Then it is easy to check that $i \circ f_v = i \circ 0$, so $f_v = 0$, whence $v = 0$, a contradiction. We conclude that $\ker i = 0$, whence $i$ is injective.

Now suppose that $i : V \to W$ is an injective linear map and that there are linear maps $L_1, L_2 : U \to V$ such that $i \circ L_1 = i \circ L_2$. Since everything in sight is a function, the argument from the second paragraph of the proof of Proposition 4.2 goes through, and we may conclude that $L_1 = L_2$ so $i$ is monic. □

We now consider monomorphisms in the category of topological spaces and continuous functions.

**Proposition 4.4.** *The monomorphisms in* Top *are precisely the injective continuous functions.*

*Proof.* Suppose that $i : X \hookrightarrow Y$ is a monomorphism in Top. Endow the singleton set $\{*\}$ with the discrete topology (which is in fact the unique topology on this set). Noting that every function $\{*\} \to X$ is continuous (check this!), we may directly adapt the proof of Proposition 4.2 to determine that $i$ is injective. It is similarly easy to see that injective continuous functions are monomorphisms. □

Given a subset $A$ of a topological space $X$, we may now ask when the inclusion $i : A \to X$ taking $a \mapsto a$ is a monomorphism. Obviously, this map is injective, so we are really asking what topologies on $A$ make $i$ continuous. Given $U \subseteq X$, we have $i^{-1}U = U \cap A$, so it suffices that $U \cap A$ be open in $A$ for all open $U \subseteq X$. In fact, it is easy to check that

$$\tau_A = \{U \cap A \mid U \subseteq X \text{ open}\}$$

is a topology on $A$. We conclude that for any topology on $A$ finer than $\tau_A$, $i : A \hookrightarrow X$ is a monomorphism. Equivalently, $\tau_A$ is the coarsest topology on $A$ such that $i : A \hookrightarrow X$ is continuous. Since $\tau_A$ satisfies such a special property, we will give it a name.

**Definition 4.5.** For a subset $A$ of a topological space $X$, the topology $\tau_A = \{U \cap A \mid U \subseteq X \text{ open}\}$ is called the *subspace topology* on $A$.

We use the subspace topology in making the following definition.

**Definition 4.6.** For topological spaces $X$, $Y$, an *embedding* $f : X \to Y$ is a continuous function which is a homeomorphism onto its image. More precisely, the induced function $f : X \to f(X)$ is a homeomorphism where the image $f(X)$ is given the subspace topology in $Y$.

It is good to think of embeddings as ways of identifying spaces with subspaces of other spaces. Note that the embedding is strictly more information than its domain: it tells us not only the homeomorphism type of the domain, but also how the domain "sits in" the codomain. Knots probably form the most famous example of a class of embeddings. Let $S^1$ denote the unit circle in $\mathbb{R}^2$ with the subspace topology. To a topologist, a *knot* is an embedding $S^1 \hookrightarrow \mathbb{R}^3$. One can think of this as a knotted string with its ends fused in three-dimensional euclidean space.

The following proposition exhibits that the notion of an embedding is not the same as that of a monomorphism in Top.

**Proposition 4.7.** *In the category* Top, *every embedding is a monomorphism, but not every monomorphism is an embedding.*

*Proof.* Suppose that $f : X \to Y$ is an embedding. Then $f : X \to f(X)$ is a homeomorphism and hence a bijection, whence $f$ must be injective. The function $f$ is continuous by hypothesis, so $f$ is a monomorphism by Proposition .

To see that the converse is false, let $X = \{0, 1\}$ with the discrete topology and let $Y = \{0, 1, 2\}$ with the indiscrete topology. (Thus all subsets of $X$ are open, while only $\varnothing$ and $Y$ are open in $Y$.) Let $f : X \hookrightarrow Y$ be the obvious inclusion, which is clearly continuous and hence a monomorphism.

Note, though, that $f : X \to f(X)$ is not open since $\{0\}$ is open in $X$ but not open in $f(X)$ with the subspace topology. Hence, by Proposition 2.7, $f$ is not an embedding. □

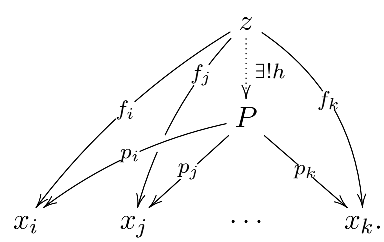
## 5. General products

In §3 we studied categorical products $x \times y$ of two objects $x$ and $y$ in a category $\mathscr{C}$. What if we have a more general collection of objects $x_i \in \mathrm{Ob}\,\mathscr{C}$ where $i$ runs through some indexing set $I$? Is there a universal object which maps to all the $x_i$? If there is, this object (along with its morphisms to the $x_i$) is called the product of the $x_i$. We develop this notion in this section.

**Definition 5.1.** Suppose that $I \to \mathrm{Ob}\,\mathscr{C}$ is a function taking $i \mapsto x_i$, $P \in \mathrm{Ob}\,\mathscr{C}$, and $I \to \mathrm{Mor}\,\mathscr{C}$ is a function taking $i \mapsto p_i \in \mathscr{C}(P, x_i)$ such that for any $z \in \mathrm{Ob}\,\mathscr{C}$ and morphisms $f_i \in \mathscr{C}(z, x_i)$, there is a unique morphisms $h \in \mathscr{C}(z, P)$ such that $p_i \circ h = f_i$ for each $i \in I$. Then the data $(P, \{p_i \mid i \in I\})$ is called the *product* of the $x_i$ in $\mathscr{C}$ and is denoted

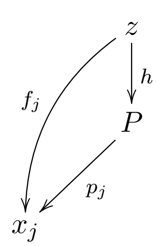
$$P = \prod_{i \in I} x_i.$$

We can express this definition succinctly via the following commutative diagram:




Some comments on the diagram are in order:

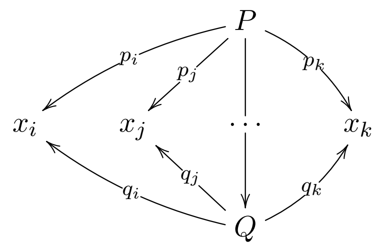
- » We read the diagram as saying that given $z$ and the morphisms $f_i$, $i \in I$, there is a unique morphism $h$ making the diagram commute.
- » We have permitted ourselves the liberty of placing morphism labels "within" the arrows they are labeling. This is strictly for aesthetic reasons. Having the arrow $f_j$ pass "under" $p_j$ is an aesthetic choice as well.
- » Each of $i$, $j$, and $k$ are elements of $I$, and the ellipses in the bottom row represent the additional $x_\ell$ belonging to $\{x_i \mid i \in I\}$. The morphisms $f_\ell$ and $p_\ell$ are implicit in the diagram as well.
- » Make sure that you can spot the $|I|$-many commuting triangles $p_i \circ h = f_i$ in the above diagram. (Of course, $|I| - 3$ of them are implicit.) In fact, we could reinterpret the above diagram as saying that there is a unique $h : z \to P$ making the diagrams




commute for each $j \in I$.

As with binary products in an arbitrary category $\mathscr{C}$, general products may or may not exist. It is the case, though, that when a general categorical product exists, it is unique up to unique isomorphism.

**Theorem 5.2.** *Suppose $(P, \{p_i \mid i \in I\})$ and $(Q, \{q_i \mid i \in I\})$ both satisfy Definition 5.1. Then there is a unique morphism $P \to Q$ making the diagram*




*commute, and this morphism is an isomorphism.*

*Proof.* The proof is essentially identical to the proof of Proposition 3.2, just with $|I| - 2$ more morphisms in play. The reader should convince herself that this is the case, carefully writing out the diagram chase if necessary. □

*Remark* 5.3. We could just as easily have defined

$$\prod A = \prod_{x \in A} x$$

for $A$ any subset of $\operatorname{Ob} \mathscr{C}$. Such a product is equipped with maps $p_x : \prod A \to x$ for each $x \in A$.

Certain products have special names and special notation:

» If $A = \{x, y\}$, then, when it exists, $\prod A$ is typically denoted $x \times y$. This matches our notion of binary product from Section 3.

» If $A = \{x_1, x_2, \ldots, x_n\}$, then, when it exists, we write $x_1 \times x_2 \times \cdots \times x_n$ or $\displaystyle\prod_{i=1}^{n} x_i$ for $\prod A$.

» If $A = \{x\}$ is a singleton set, it is easy to check that

$$\prod \{x\} = x$$

where the morphism $p_x : x \to x$ is the identity morphism $1_x$. This product always exists.

» If $A = \varnothing$ is the empty subset of $\operatorname{Ob} \mathscr{C}$, then, when it exists, $\prod \varnothing$ is called the *terminal object* of $\mathscr{C}$, and is typically denoted $*$ (or sometimes 1). If we carefully sort out the quantifiers in Definition 5.1, we see that $*$ is an object in $\mathscr{C}$ which admits a unique morphism from each $z \in \operatorname{Ob} \mathscr{C}$. In other words, $|\mathscr{C}(z, *)| = 1$ for each $z \in \operatorname{Ob} \mathscr{C}$. The category $\mathrm{Set}$ of sets and functions has terminal object any singleton set. Similarly, $\mathrm{Top}$ has terminal object any single point topological space. The category $\mathrm{Vect}_k$ of $k$-vector spaces has terminal object $0 = \{0\}$.

We now undertake the task of identifying products in several familiar categories. We begin with $\mathrm{Set}$.

**Definition 5.4.** Given a family $\{X_i \mid i \in I\}$ of sets $X_i$ (in other words, a "function" $I \to \operatorname{Ob} \mathrm{Set}$, ignoring the fact that $\operatorname{Ob} \mathrm{Set}$ is not a set), define the *cartesian product*

$$\bigtimes_{i \in I} X_i$$

to be the set of functions $x : I \to \bigcup_{i \in I} X_i$ such that $x(i) \in X_i$. In a cartesian product, we will typically write $x_i$ for $x(i)$. An element of $\bigtimes_{i \in I} X_i$ is called a *tuple*, and we will frequently write $(x_i)$ or $(x_i)_{i \in I}$ for $x$.

Given $j \in I$, we define the *projection* $p_j : \bigtimes_{i \in I} X_i \to X_j$ by the formula $p_j(x) = x_j$.

**Proposition 5.5.** *The cartesian product* $\bigtimes_{i\in I} X_i$ *along with its projection functions* $p_i$ *form the product* $\prod_{i\in I} X_i$ *in* Set.

*Proof.* Given a set $Z$ and functions $f_i : Z \to X_i$ for $i \in I$, define $h : Z \to \bigtimes X_i$ by $h(z) = (f_i(z))_{i\in I}$. Then $p_i(h(z)) = f_i(z)$ for all $i \in I$ and all $z \in Z$, whence $p_i \circ h = f_i$ for all $i \in I$. The reader may check that the equations $p_i \circ h = f_i$ completely specify $h$, so $\bigtimes X_i$ is in fact the product in Set of the $X_i$. □

*Remark* 5.6. Now that we have proven that cartesian and categorical products are identical in Set, we will adopt the more common notation $\prod X_i$ for $\bigtimes X_i$ when the $X_i$ are sets.

**Example 5.7.** For $k$ a field, the product in $\mathrm{Vect}_k$ of $k$-vector spaces $V_i$, $i \in I$, has underlying set the cartesian product of the $V_i$ with linear structure given by $(v_i) + (w_i) = (v_i + w_i)$ and $\lambda(v_i) = (\lambda v_i)$. The projection maps are usual ones. The reader may easily check the correctness of these assertions.

Note, though, that $\mathrm{FinVect}_k$, the category of finite-dimensional $k$-vector spaces, does not have arbitrary products. Indeed, $\prod_{i\in I} V_i$ exists if and only if $\sum_{i\in I} \dim_k V_i < \infty$; in this case, the product is the same as that for $\mathrm{Vect}_k$.

As we shall presently see, arbitrary products do exist in Top, the category of topological spaces and continuous functions. Rather than leap directly to the definition/theorem (*defineorem*?) giving these products, we will gradually develop the necessary concepts.

For topological spaces $X_i$, $i \in I$, it is reasonable to guess that the set underlying $\prod_{i\in I} X_i$ is the cartesian product of the sets underlying each $X_i$. If this is the case, then we need each of the projection functions $p_j : \prod X_i \to X_j$ to be continuous. For $U \subseteq X_j$, the preimage $p_j^{-1}U$ is exactly $\prod X_i$ but with the $X_j$ factor replaced by $U$. In other words,

$$p_j^{-1}U = \{(x_i) \mid x_i \in X_i \text{ for } i \in I \setminus \{j\} \text{ and } x_j \in U\} \subseteq \prod X_i$$

is the set of tuples with $j$-th coordinate in $U$. When $U$ is open in $X_j$, this set must be open in $\prod X_i$. Let

$$\mathscr{S}_j = \{p_j^{-1}U \mid U \subseteq X_j \text{ open}\}$$

be the set of such subsets of $\prod X_i$, and let

$$\mathscr{S} = \bigcup_{j\in I} \mathscr{S}_j.$$

Supposing that $\tau$ is the topology on $\prod X_i$ which we seek (*i.e.*, the topology which will make $\prod X_i$ a product in Top), we see that we must have $\mathscr{S} \subseteq \tau$. Note that $\mathscr{S}$ is a subbasis. (Indeed, $p_j^{-1}X_j = \prod X_i$ is already an element of $\mathscr{S}$, and the subbasis condition only mandates that the union of all elements of $\mathscr{S}$ be $\prod X_i$.)

Since little else seems to be hiding in Definition 5.1, it is reasonable to guess that $\tau$ is the topology $\tau_{\mathscr{S}}$ generated by the subbasis $\mathscr{S}$. This is in fact the case. We will develop some properties of $\tau_{\mathscr{S}}$ and then prove that $(\prod X_i, \tau_{\mathscr{S}})$ along with the projection maps $p_j : \prod X_i \to X_j$ form a product in Top.

**Proposition 5.8.** *For* $J \subseteq I$ *and* $U_j \subseteq X_j$ *open for each* $j \in J$, *let* $\mathscr{U} = \{U_j \mid j \in J\}$ *and let*

$$B_{\mathscr{U}} = \prod_{i\in I} V_i$$

*where* $V_j = U_j$ *if* $j \in J$ *and* $V_i = X_i$ *if* $i \in I \setminus J$. *We will refer to* $J$ *as the* index subset *of* $\mathscr{U}$. *Using this terminology, the basis* $\mathscr{B}$ *generated by* $\mathscr{S}$ *is equal to*

$$\mathscr{B} = \{B_{\mathscr{U}} \mid \mathscr{U} \text{ has finite index set}\}.$$

*As such, the topology generated by* $\mathscr{S}$ *consists of arbitrary unions of elements of* $\mathscr{B}$.

*Remark* 5.9. Note that the basis sets $B_{\mathscr{U}}$ are precisely products of open subsets $U_i$ of each $X_i$ where all but finitely many $U_i = X_i$. Later, we will comment on *box topology*, a different topology on the cartesian product of the $X_i$ in which we permit the $U_i$ to be arbitrary open subsets of $X_i$.

*Proof.* The basis $\mathscr{B}$ generated by $\mathscr{S}$ consists of finite intersections of sets in $\mathscr{S}$. Note that

$$p_j^{-1}U \cap p_j^{-1}V = p_j^{-1}(U \cap V),$$

so it suffices to consider finite intersections

$$B = p_{j_1}^{-1}U_{j_1} \cap p_{j_2}^{-1}U_{j_2} \cap \cdots \cap p_{j_n}^{-1}U_{j_n}$$

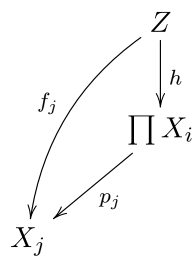
in which the indices $j_k$ are all distinct.

A point $(x_i)$ is in $B$ if and only if $x_{j_1} \in U_{j_1}, x_{j_2} \in U_{j_2}, \ldots, x_{j_n} \in U_{j_n}$ while the other $x_i$ are allowed to range freely through $X_i$. Hence $B = B_{\mathscr{U}}$ where $\mathscr{U} = \{U_{j_1}, \ldots, U_{j_n}\}$, completing our proof. □

We now come to our main theorem for this section, which identifies products in Top.

**Theorem 5.10.** *For topological spaces $X_i$, $i \in I$, the cartesian product $\prod_{i \in I} X_i$ equipped with the topology $\tau_{\mathscr{S}}$ generated by the subbasis $\mathscr{S}$ is the product of the $X_i$ in* Top.

*Proof.* We aim to prove that for any space $Z$ and collection $\{f_i : Z \to X_i \mid i \in I\}$ of continuous functions, there is a unique continuous function $h : Z \to \prod X_i$ such that the diagrams




commute for each $j \in I$. Given such data, define $h$ by the formula

$$h(z) = (f_i(z))_{i \in I}.$$

In our proof of Proposition 5.5, we have already seen that $h$ is the unique *function* making the diagram commute. It remains to show that $h$ is continuous relative to $\tau_{\mathscr{S}}$.

Since $\tau_{\mathscr{S}}$ is generated by the subbasis $\mathscr{S}$, it suffices to check that each of the subbasic opens $p_j^{-1}U$ for $j \in I$ and $U \subseteq X_j$ open are taken to open subsets of $Z$ by $h^{-1}$. Since $p_j \circ h = f_j$, we have $h^{-1} \circ p_j^{-1} = f_j^{-1}$. The function $f_j$ is continuous by hypothesis, so $h^{-1}(p_j^{-1}U) = f_j^{-1}U$ is open in $Z$, as desired. □

The following theorem is really just a reinterpretation of the statement that $\prod X_i$ is a categorical product in Top. In order to state it, we introduce a small amount of terminology. Given a function (not necessarily continuous) $f : Z \to \prod X_i$, define the *component functions* of $f$ to be $f_i = p_i \circ f$ for $i \in I$. Also note that given functions $f_i : Z \to X_i$, $i \in I$, Proposition 5.5 tells us that there is a unique function $f : Z \to \prod X_i$ whose component functions are $f_i$. (It makes no difference to call the function $f$ instead of $h$.)

**Theorem 5.11.** *A function $f : Z \to \prod X_i$ is continuous if and only if each of its component functions $f_i = p_i \circ f$ is continuous.*

*Proof.* First suppose that $f$ is continuous. Since each $p_i$ is continuous and compositions of continuous functions are continuous, we learn that $f_i = p_i \circ f$ is continuous for each $i \in I$.

Now suppose that each $f_i$ is continuous. By Proposition 5.5, $f$ is the unique function making $f_i = p_i \circ f$ for each $i \in I$, and the fact that $\prod X_i$ is a product in $\mathrm{Top}$ now guarantees that $f$ is continuous. $\square$

Given Theorem 5.10, it is reasonable to make the following definition.

**Definition 5.12.** The topology on $\prod_{i\in I} X_i$ generated by the subbasis $\mathscr{S}$ is called the *product topology* on $\prod X_i$. If we make no further comment, we will always assume that $\prod X_i$ has been given the product topology.

There is another, more naïve and less useful topology on $\prod X_i$ which we will sometimes consider.

**Definition 5.13.** Given a cartesian product $\prod_{i\in I} X_i$ of topological spaces, let $\mathscr{B}_{\text{box}}$ denote the collection of sets $B_{\mathscr{U}}$ (defined in Proposition 5.8) in which the index set of $\mathscr{U}$ is allowed to be any subset of $I$. The *box topology* on $\prod X_i$ is the topology generated by this basis.

*Remark* 5.14. We leave it to the reader to check that $\mathscr{B}_{\text{box}}$ is in fact a basis. Also note that we could just as well mandate that the index set of $\mathscr{U}$ be all of $I$.

*Remark* 5.15. If the index set $I$ is finite, then the box and product topologies are the same.

The following example is one of those examples that every topologist knows. You should commit it to memory so that you never make the mistake of confusing the product and box topologies.

**Example 5.16.** Let $\mathbb{R}^{\mathbb{N}}$ denote the countably infinite product of $\mathbb{R}$ with itself, *i.e.*,

$$\mathbb{R}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{R}.$$

Let $\Delta$ denote the diagonal function

$$\begin{aligned} \Delta : \mathbb{R} &\longrightarrow \mathbb{R}^{\mathbb{N}} \\ t &\longmapsto (t, t, t, \ldots). \end{aligned}$$

Each component function $\Delta_n$ of $\Delta$ is the identity function $1_{\mathbb{R}}$, hence $\Delta$ is continuous if we give $\mathbb{R}^{\mathbb{N}}$ the product topology.

Now consider $\mathbb{R}^{\mathbb{N}}$ with the box topology. The set

$$B = \prod_{n\in\mathbb{N}} \left( \frac{-1}{n+1}, \frac{1}{n+1} \right) = (-1,1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is an element of $\mathscr{B}_{\text{box}}$ and hence is a box-open subset of $\mathbb{R}^{\mathbb{N}}$. Note, though, that

$$\Delta^{-1} B = \bigcap_{n\in\mathbb{N}} \left( \frac{-1}{n+1}, \frac{1}{n+1} \right) = \{0\}.$$

(*Exercise*: check that the above equalities hold!) Hence $\Delta^{-1}B$ is *not* an open subset of $\mathbb{R}$, and we conclude that $\Delta$ is not continuous relative to the box topology on $\mathbb{R}^{\mathbb{N}}$.

We can draw the following morals from this example:

(a) The product and box topologies are in fact different topologies in general (despite the fact that they are the same when the index set is finite). We know this because we have just seen that they have different sets of continuous functions. From this, we can conclude that there are instances in which the box topology is strictly finer than the product topology.
(b) The fact that the product topology makes $\prod X_i$ into a categorical product should convince us of its superiority to the box topology. But even if we do not trust category theorists, continuity of $\Delta$ is clearly a desirable property, and we must use the product topology to guarantee this.

## 6. QUOTIENTS

Recall the notion of an *equivalence relation* $\sim$ on a set $X$. Specifically, $\sim$ is a relation which is reflexive ($x \sim x$ for all $x \in X$), symmetric ($x \sim y \implies y \sim x$), and transitive ($x \sim y,\ y \sim z \implies x \sim z$). For instance, the relation

$$x \sim y \iff x - y \in \mathbb{Z}$$

is an equivalence relation on $\mathbb{R}$.

Given an equivalence relation $\sim$ on $X$, we may partition $X$ into *equivalence classes* (relative to $\sim$). The equivalence class of $x \in X$ is the set of $y \in X$ such that $x \sim y$. Writing $X/\sim$ for the set of $\sim$ equivalence classes, we see that

$$X = \coprod_{A \in X/\sim} A$$

where $\amalg$ denotes disjoint union. In the case of the example from the previous paragraph, we see that $\mathbb{R}/\sim = \{a + \mathbb{Z} \mid a \in [0,1)\}$ and the partition into equivalence classes takes the form

$$\mathbb{R} = \coprod_{a \in [0,1)} a + \mathbb{Z}.$$

Not wishing to be *circular* in our reasoning, we will postpone commenting on the topology of $\mathbb{R}/\sim$ until we have introduced the quotient topology and quotient spaces.

Given a partition

$$X = \coprod_{i \in I} X_i$$

of a set $X$, there is an equivalence relation given by $x \sim y$ if and only if there is an index $i \in I$ such that $x, y \in X_i$. For this equivalence relation, $X/\sim = \{X_i \mid i \in I\}$.

There is a natural *quotient function* $q : X \to X/\sim$ taking $x$ to the equivalence class of $x$. Note that $q$ is surjective. Given any surjective function $p : X \to Y$, we can partition $X$ into the *fibers* of $p$, $p^{-1}\{y\}$ for $y \in Y$. This partition yields an equivalence relation whose associated quotient function is $p$. As such, we see that equivalence relations, partitions, and surjective functions are three sides of the same hypercoin.

Now suppose that $X$ is a topological space with an equivalence relation $\sim$ on its underlying set. How should we topologize $X/\sim$? Our search should be guided by the quotient map $q : X \to X/\sim$, which ought to be continuous. Consider the set

$$\tau = \{U \subseteq X/\sim \mid q^{-1}U \text{ is open in } X\}.$$

The reader may check that $\tau$ is a topology, and is in fact the finest topology on $X/\sim$ such that $q$ is continuous. We call $\tau$ the *quotient topology* on $X/\sim$.

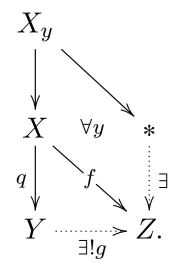
It is not the case that every surjective continuous function $p : X \to Y$ arises from an equivalence relation on $X$. Indeed, the quotient map $q : X \to X/\sim$ has the special property that $U \subseteq X/\sim$ is open if and only if $q^{-1}U$ is open in $X$. Meanwhile, for $p$ to be continuous, we only need that $U$ open in $Y$ implies that $p^{-1}U$ is open in $X$. But this crucial distinction is the only one. If we want to recover our hypercoin in the topological setting, we must replace "surjective function" with the following notion of "quotient map".

**Definition 6.1.** A continuous function $q : X \to Y$ is a *quotient map* if it is surjective and $U \subseteq Y$ is open if and only if $q^{-1}U \subseteq X$ is open. If $q$ is a quotient map, then $Y$ is called a *quotient* of $X$ or a *quotient space*. If $q$ is the quotient function of an equivalence relation $\sim$, then we will always give $Y = X/\sim$ the *quotient topology*, the unique topology that makes $q$ a quotient map.

Note that this is not a strictly categorical definition. While it is possible to recast quotients in purely categorical terms, there is little benefit to doing so (at least currently), and we find the above derivation sufficient for our purposes. Nonetheless, it is the case that quotient maps $q : X \to Y$ satisfy a certain universal property.

**Theorem 6.2.** *Given a quotient map $q : X \to Y$ and $y \in Y$, let $X_y := q^{-1}\{y\}$ denote the* fiber *of $q$ over $y$. Suppose $f : X \to Z$ is a continuous function which is constant on $X_y$ for each $y \in Y$. Then there is a unique map $g : Y \to Z$ such that $g \circ q = f$. If, additionally, $f$ is a quotient map with $f^{-1}\{f(x)\} = X_{q(x)}$ for all $x \in X$, then $g$ is a homeomorphism.*

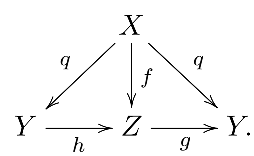
We may diagrammatically represent the first part of the theorem as follows:



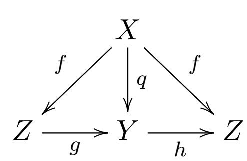

Here $*$ denotes a one point space, $X_y \to X$ is the inclusion mapping, and the commutativity of the upper right parallelogram says that $f$ is constant on $X_y$ (since the image of $* \to Z$ is necessarily a single point). The quantifiers $\forall y$ and $\exists$ (on the right-hand vertical arrow) are meant to indicate that for every $y \in Y$ there is a map $* \to Z$ such that the upper right parallelogram commutes, recapitulating the hypothesis that $f$ is constant on each fiber. In this case, there is a unique map $g : Y \to Z$ making the bottom triangle commute.

*Proof of Theorem 6.2.* Cf. Theorem 22.2 in Munkres for the first part of the theorem. It remains to show that $g$ is a homeomorphism under the additional hypotheses on $f$. We begin by showing that the hypotheses of the first part of the theorem hold with $f$ and $q$ swapped. Since both $q$ and $f$ are surjective, it is clear that the fibers of $q$ are in bijective correspondence with the fibers of $f$. Since $q$ is constant on its own fibers, we see that it is also constant on the fibers of $f$. Hence there is a unique map $h : Z \to Y$ such that $h \circ f = q$.

We now check that $g$ is a homeomorphism by proving that $g \circ h = 1_Y$ and $h \circ g = 1_Z$. Consider the commutative diagram




Thus, by the first part of the theorem (with $q$ playing the roles of both $q$ and $f$), we see that $g \circ h$ is the unique map $Y \to Y$ such that $(g \circ h) \circ q = q$. Since $1_Y$ is a map satisfying $1_Y \circ q = q$, we learn that $g \circ h = 1_Y$. A similar argument with the commutative diagram




shows that $h \circ g = 1_Z$, concluding our proof. □

We now return to $\mathbb{R}/\sim$ where $x \sim y \iff x - y \in \mathbb{Z}$. Note that each equivalence class is of the form $a + \mathbb{Z}$ for exactly one $a \in [0, 1)$, and $1 + \mathbb{Z} = \mathbb{Z}$, so it seems like $\mathbb{R}/\sim$ circles back on itself when $a$ goes past 1. This is precisely the case, as we shall now prove via Theorem 6.2.

Let $q : \mathbb{R} \to \mathbb{R}/\sim$ denote the quotient map, and consider $f : \mathbb{R} \to S^1$ taking $a \mapsto e^{2\pi i a}$. Here $S^1$ is the unit circle in $\mathbb{C} \cong \mathbb{R}^2$ with the standard (*e.g.* subspace) topology. The reader may check that $f$ is a continuous map. The fibers of $q$ are precisely the equivalence classes $a + \mathbb{Z}$, and we see that for $b \in \mathbb{Z}$, $f(a+b) = e^{2\pi i(a+b)} = e^{2\pi i a}$. Thus $f$ is constant on the fibers of $q$ and we conclude that there is a unique map $g : \mathbb{R}/\sim \to S^1$ such that $g \circ q = f$.

We now check the additional hypotheses guaranteeing that $g$ is a homeomorphism. The map $f$ is clearly surjective, and the reader may check that $f$ is in fact a quotient map. (The details are painful to write down, but more or less obvious.) Moreover, $f^{-1}\{f(a)\} = a + \mathbb{Z} = q^{-1}\{q(a)\}$ for all $a \in \mathbb{R}$. Thus the second part of Theorem 6.2 tells us that $g$ is a homeomorphism; *i.e.* $\mathbb{R}/\sim$ is a circle.

*Remark* 6.3. If you have seen group-theoretic quotients before, you may recognize $\mathbb{R}/\sim$ as $\mathbb{R}/\mathbb{Z}$ where $\mathbb{Z}$ is considered as a subgroup of the additive group $\mathbb{R}$. If you feel comfortable with this, I highly encourage you to do the supplementary exercises on topological groups on pp.145–146 of Munkres.

## 7. Functors

Category theory tells us that in order to understand mathematical objects, we must understand the relations (*i.e.* morphisms) between those objects. If categories themselves are mathematical objects, then category theory demands that we must study how categories are related. Of course, this is both possible and useful: a morphism of categories is called a *functor*.

**Definition 7.1.** If $\mathscr{C}$ and $\mathscr{D}$ are categories, then a *functor* $F : \mathscr{C} \to \mathscr{D}$ consists of assignments

- $F_{\mathrm{Ob}} : \mathrm{Ob}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{D}$, and
- $F_{\mathrm{Mor}} : \mathscr{C}(a,b) \to \mathscr{C}(F_{\mathrm{Ob}}a, F_{\mathrm{Ob}}b)$ for all $a, b \in \mathrm{Ob}\,\mathscr{C}$

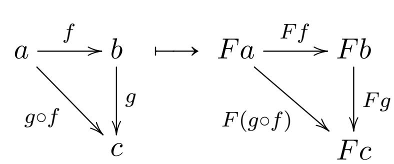
satisfying the following properties:

- $F_{\mathrm{Mor}} 1_a = 1_{F_{\mathrm{Ob}}a}$ for all $a \in \mathrm{Ob}\,\mathscr{C}$, and
- $F_{\mathrm{Mor}}(g \circ f) = (F_{\mathrm{Mor}} g) \circ (F_{\mathrm{Mor}} f)$ for all composable morphisms $f, g \in \mathrm{Mor}\,\mathscr{C}$.

*Remark* 7.2. In practice, the distinction between $F_{\mathrm{Ob}}$ and $F_{\mathrm{Mor}}$ is obvious from context, and we simply write $Fa = F_{\mathrm{Ob}}a$ when $a \in \mathrm{Ob}\,\mathscr{C}$ and $Ff = F_{\mathrm{Mor}}f$ when $f \in \mathrm{Mor}\,\mathscr{C}$. Abusing notation in this fashion, we see that properties satisfied by $F$ take the simpler forms:

- $F1_a = 1_{Fa}$, and
- $F(g \circ f) = (Fg) \circ (Ff)$.

It is nice to think of the second property as saying that $F$ takes commutative diagrams in $\mathscr{C}$ to commutative diagrams in $\mathscr{D}$:




We have encountered many functors in the past, and have even used functorial properties in arguments. Here are a few pertinent examples.

**Example 7.3.** Given sets $A$ and $B$ and a function $f : A \to B$, recall the direct image function $f_* : 2^A \to 2^B$ and preimage function $f^* : 2^B \to 2^A$. You proved that these satisfied the following compatibilities:

$$(g \circ f)_* = g_* \circ f_* \qquad \text{and} \qquad (g \circ f)^* = f^* \circ g^*$$

whenever the composition of functions $g \circ f$ makes sense. Additionally, $(1_A)_* = 1_{2^A} = (1_A)^*$. Thus we may define a functor $F : \mathrm{Set} \to \mathrm{Set}$ via $FA = 2^A$ and $Ff = f_*$, the direct image functor.

We *almost* have a preimage functor $G$ as well, given by defining $GA = 2^A$ and $Gf = f^*$, but $G$ "reverses the direction" of morphisms, and $G(g \circ f) = Gf \circ Gg$ instead of $Gg \circ Gf$. (In fact, the latter composition does not even make sense in general.) Such an assignment gets a name as well.

**Definition 7.4.** If $\mathscr{C}$ and $\mathscr{D}$ are categories, then $G$ is a *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ if it consists of assignments

» $G : \mathrm{Ob}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{D}$, and
» $G : \mathscr{C}(a, b) \to \mathscr{D}(Gb, Ga)$ for all $a, b \in \mathrm{Ob}\,\mathscr{C}$

such that

» $G1_a = 1_{Ga}$ for all $a \in \mathrm{Ob}\,\mathscr{C}$, and
» $G(g \circ f) = Gf \circ Gg$ for all composable morphisms $f, g \in \mathrm{Mor}\,\mathscr{C}$.

We write $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ when $G$ is a contravariant functor. In contradistinction, functors $F : \mathscr{C} \to \mathscr{D}$ are sometimes called *covariant functors*.

Clearly, the assignment $G : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ given by $GA = 2^A$, $Gf = f^*$ is a contravariant functor from $\mathrm{Set}$ to $\mathrm{Set}$.

*Remark* 7.5. Secretly, $\mathscr{C}^{\mathrm{op}}$ is notation for the *opposite category* of $\mathscr{C}$. The category $\mathscr{C}^{\mathrm{op}}$ has the same objects and morphisms as $\mathscr{C}$, but the source and target functions are swapped. The categorically inclined reader should spend some time verifying that $\mathscr{C}^{\mathrm{op}}$ is a category, and checking that a contravariant functor from $\mathscr{C}$ to $\mathscr{D}$ is the same thing as a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

**Example 7.6.** Many familiar categories $\mathscr{C}$ have a *forgetful functor* $U : \mathscr{C} \to \mathrm{Set}$. For instance, the forgetful functor $U : \mathrm{Top} \to \mathrm{Set}$ takes a space $X$ to its underlying set $UX = X$ and takes a continuous function $f : X \to Y$ to the function $Uf = f : UX \to UY$. It should be clear that this satisfies the functor properties.

The letter $U$ stands for *underlying*, and we have such a forgetful functor basically whenever the objects in a category are sets equipped with extra structure and the morphisms of a category a special types of functions. There are forgetful functors from groups to sets, from rings to sets, from vector spaces to sets, *etc*.

There are also intermediate forgetful functors. For instance, every ring has an underlying (additive) abelian group. Since ring homomorphisms are special types of group homomorphisms on the underlying abelian groups, we get a forgetful functor $U : \mathrm{Ring} \to \mathrm{AbGp}$. If we then forget all the way to sets, we get the bizarre equality of functors $U \circ U = U$ (where each $U$ is denoting a different forgetful functor).

**Example 7.7.** Consider the category of vector spaces $\mathrm{Vect}_k$ over a field $k$. Each $k$-vector space $V$ has a linear dual $V^\vee = \mathrm{Vect}_k(V, k)$ (where $k$ is given the standard $k$-linear structure). Each $k$-linear map $f : V \to W$ has a dual $f^\vee : W^\vee \to V^\vee$ given by $f(\alpha) = \alpha \circ f$. It is a standard exercise in linear algebra to check that $(\ )^\vee$ is a functor $\mathrm{Vect}_k^{\mathrm{op}} \to \mathrm{Vect}_k$.

**Example 7.8.** The dual vector space functor is our first example of a *representable functor*. Given a (locally small[1]) category $\mathscr{C}$ and object $x \in \mathrm{Ob}\,\mathscr{C}$, there is a functor $\mathscr{C}(\ , x) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ taking objects $a$ to the set of morphisms $\mathscr{C}(a, x)$ and morphisms $f : a \to b$ to $\mathscr{C}(f, x) : \mathscr{C}(b, x) \to \mathscr{C}(a, x)$ such that $\mathscr{C}(f, x)(\alpha) = \alpha \circ f$. We call $\mathscr{C}(\ , x)$ the functor *reprsented* by $x$. The dual vector space functor is the functor represented by the one-dimensional $k$-vector space $k$.

**Example 7.9.** The categorically inclined reader should define and investigate the *corepresentable functors* $\mathscr{C}(x,\ )$ for $x \in \mathrm{Ob}\,\mathscr{C}$.

[1]Locally small categories are categories $\mathscr{C}$ for which $\mathscr{C}(a, b)$ is a set (as opposed to a class or some other set-theoretic monster) for each $a, b \in \mathscr{C}$.

**Example 7.10.** Recall that for a space $X$, $\pi_0 X$ denotes the path-connected components of $X$. In your homework, you will prove that $\pi_0$ is a functor $\mathrm{Top} \to \mathrm{Set}$. The categorically inclined reader may define the category $\mathrm{Hot}$ of topological spaces and homotopy classes of maps between spaces, and then prove that $\pi_0$ is the functor $\mathrm{Hot} \to \mathrm{Set}$ corepresented by the unit interval $I = [0, 1]$.

7.1. **The fundamental group functor.** We now turn to our motivation for introducing the language of functors: the fundamental group functor. We have already noted that the fundamental group depends on a choice of basepoint, so we will need to introduce a category of "topological spaces with basepoint" in order to develop our story. We leave it as an easy exercise to the reader to check that the following choice of objects and morphisms does indeed define a category.

**Definition 7.11.** The category $\mathrm{Top}_*$ of *based topological spaces* has objects which are pairs $(X, x_0)$ where $X$ is a topological space and $x_0 \in X$ is a point of $X$. A morphism $(X, x_0) \to (Y, y_0)$ in $\mathrm{Top}_*$ is a continuous function $f : X \to Y$ such that $f(x_0) = y_0$.

**Theorem 7.12.** *Given a based space $(X, x_0)$, let $\pi_1(X, x_0)$ denote the fundamental group of $X$ based at $x_0$. Given a based map $f : (X, x_0) \to (Y, y_0)$, let $\pi_1 f$ denote the assignment*

$$\begin{aligned} \pi_1(X, x_0) &\longrightarrow \pi_1(Y, y_0) \\ [\gamma : I \to X] &\longmapsto [f \circ \gamma]. \end{aligned}$$

*Then $\pi_1$ is a functor $\mathrm{Top}_* \to \mathrm{Gp}$.*

*Proof sketch.* We must check the following:

(1) The assignment $\pi_1 f : [\gamma] \mapsto [f \circ \gamma]$ is well-defined (*i.e.* it does not depend on our choice of path homotopy class representatives).
(2) The function $\pi_1 f$ is in fact a group homomorphism.
(3) If $1_{(X,x_0)}$ is the identity map, then $\pi_1 1_{(X,x_0)}$ is the identity homomorphism on $\pi_1(X, x_0)$.
(4) If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ are based maps, then $\pi_1(g \circ f) = \pi_1 g \circ \pi_1 f$.

In order to check (1), first note that $(f \circ \gamma)(0) = f(\gamma(0)) = f(x_0) = y_0$, and similarly $(f \circ \gamma)(1)$, so $[f \circ \gamma] \in \pi_1(Y, y_0)$. Now suppose that $H : \gamma \simeq_p \gamma'$. The reader may check that $f \circ H$ is a path homotopy from $f \circ \gamma$ to $f \circ \gamma'$, whence $[f \circ \gamma] = [f \circ \gamma']$. This proves that $\pi_1 f$ is well-defined.

To check (2), suppose that $[\gamma]$ and $[\delta]$ are elements of $\pi_1(X, x_0)$. The reader may check that

$$(f \circ \gamma) * (f \circ \delta) = f \circ (\gamma * \delta),$$

whence $\pi_1 f(\gamma) * \pi_1 f(\delta) = \pi_1 f(\gamma * \delta)$. We conclude that $\pi_1 f$ is a group homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$.

The easiest of the properties, (3) holds because $(\pi_1 1_{(X,x_0)})[\gamma] = [1 \circ \gamma] = [\gamma]$.

Finally, we check (4), which asserts that $\pi_1$ respects composition of based maps. Suppose that $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ are based maps. Then

$$\pi_1(g \circ f) : [\gamma] \longmapsto [(g \circ f) \circ \gamma]$$

while

$$\pi_1 g \circ \pi_1 f : [\gamma] \longmapsto [g \circ (f \circ \gamma)].$$

Since $(g \circ f) \circ \gamma = g \circ (f \circ \gamma)$, we conclude that

$$\pi_1(g \circ f) = \pi_1 g \circ \pi_1 f.$$

□