Homotopy

Facts about approximating closed curves (read pp. 141–144):

For $\gamma_1, \gamma_2 : I \to \mathbb{C}$ curves (cts fns) define

$$\|\gamma_1 - \gamma_2\| = \sup_{t \in I} |\gamma_1(t) - \gamma_2(t)|.$$

Thm 1 If $\gamma : I \to \mathbb{C}$ is a curve, then $\forall \varepsilon > 0$ there is a piecewise linear curve $\tilde{\gamma}$ s.t. $\|\tilde{\gamma} - \gamma\| < \varepsilon$.

Thm 2 If $\gamma : I \to \mathbb{C}$ closed curve, $z \in \mathbb{C} - \gamma(I)$, then $\exists \delta > 0$ s.t. if γ_1, γ_2 are paths with $\|\gamma - \gamma_j\| < \delta$, $j = 1, 2$, then $\mathrm{Ind}_{\gamma_1}(z) = \mathrm{Ind}_{\gamma_2}(z)$.

Defn If $\gamma : I \to \mathbb{C}$ closed curve, $z \in \mathbb{C} - \gamma(I)$, choose $\delta > 0$ as in Thm 2 and $\tilde{\gamma}$ as in Thm 1 (w/ $\varepsilon = \delta$). Set $\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_{\tilde{\gamma}}(z)$.

Thm $\mathrm{Ind}_\gamma(z)$ is a locally constant function of γ.

$U \subseteq \mathbb{C}$ open, $\gamma_0, \gamma_1 : I \to U$, $I = [0,1]$. closed curves

Defn γ_0, γ_1 are *homotopic* in U if $\exists$ cts $h : I^2 \to U$ s.t.

(a) $h(0,t) = \gamma_0(t) \quad \forall t \in I$

(b) $h(1,t) = \gamma_1(t) \quad \forall t \in I$

(c) $h(s,0) = h(s,1) \quad \forall s \in I$

Write $\gamma_s(t) := h(s,t)$

Thm $\forall s_0 \in I, \varepsilon > 0 \; \exists \delta > 0$ s.t. $\|\gamma_s - \gamma_{s_0}\| < \varepsilon$ for $|s - s_0| < \delta$.

Pf Cts fn on cpt set is unif cts. $\square$

<u>Thm</u> If $\gamma_0, \gamma_1 : I \to U$ are homotopic in U then

$$\mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z) \quad \forall z \in \mathbb{C} \setminus U.$$

<u>Pf</u> By previous thms, $\forall s_0 \in I \;\exists \varepsilon > 0$ s.t. $\|\gamma_s - \gamma_{s_0}\| < \varepsilon \Rightarrow \mathrm{Ind}_{\gamma_s}(z) = \mathrm{Ind}_{\gamma_{s_0}}(z)$ and $\exists \delta > 0$ s.t. $|s - s_0| < \delta \Rightarrow \|\gamma_s - \gamma_{s_0}\| < \varepsilon$. As such, $\mathrm{Ind}_{\gamma_s}(z)$ is constant on $(s_0 - \delta, s_0 + \delta)$. Thus the set (if $s \in I$) on which $\mathrm{Ind}_{\gamma_s}(z)$ takes any given value is open. Since I is connected, $\mathrm{Ind}_{\gamma_s}(z)$ is constant. $\square$

<u>Thm</u> If $\gamma_0, \gamma_1 : I \to U$ are homotopic closed paths in U, then $\Gamma = \gamma_1 - \gamma_0$ is nullhomologous in U and $\int_{\gamma_1} f = \int_{\gamma_0} f$. $\square$

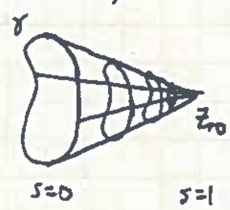
<u>e.g.</u>

are $\underbrace{\text{htpic in } \mathbb{C} - \{0\}}_{\text{explicit htpy}}$ but $\underbrace{\text{not in } \mathbb{C} - \{3/2\}}_{\text{different indices at } 3/2}$.

<u>Defn</u> A connected open set U is <u>simply connected</u> if every closed curve in U is homotopic to a point (i.e. to a constant curve) in U.

<u>Prop</u> Convex open sets are simply connected.

<u>Pf</u> $U \subseteq \mathbb{C}$ convex open, $z_0 \in U$. If $\gamma : I \to U$ closed curve in U, define $h(s,t) = (1-s)\gamma(t) + s z_0$, the "straight line homotopy."



$\square$

Thm $U \subseteq \mathbb{C}$ open and connected then TFAE:

(a) U is simply connected

(b) every cycle in U is nullhomologous

(c) $\mathbb{C} \setminus U$ has no bounded components

(d) $\forall$ cycles Γ in U and analytic fns f on U, $\int_\Gamma f = 0$

(e) every analytic fn f on U has an antiderivative

(f) every harmonic fn f on U has a harmonic conjugate

(g) every nonvanishing analytic fn f on U has an analytic logarithm

(h) every nonvanishing analytic fn f on U has an analytic square root.

Pf (a) $\Rightarrow$ (b): $Ind_*(z) = 0$ on $\mathbb{C} \setminus * \supseteq \mathbb{C} \setminus U$.

(b) $\Rightarrow$ (c): ~~If $\mathbb{C} \setminus U$ has a bdd cpt C, then C is contained in a closed bdd $A \subseteq \mathbb{C} \setminus U$ s.t. $B = (\mathbb{C} \setminus U) \setminus A$ is also closed. (Check!)~~

Idea Build a curve in U around a bdd cpt of $\mathbb{C} \setminus U$ with index 1 in the cpt.

(c) $\Rightarrow$ (d): $z \in \mathbb{C} \setminus U$ is in the unbdd cpt of $\mathbb{C} \setminus \Gamma(I)$ so $Ind_\Gamma(z) = 0$ ~~analog's integral~~, so Cauchy's Thm gives $\int_\Gamma f = 0$.

(d) $\Rightarrow$ (e): $g(z) = \int_{\gamma_z} f(w)\,dw$ for γ_z any path z_0 to z is an antideriv.
Fix $z_0 \in U$.

(e) $\Rightarrow$ (f): Read pf of Thm 3.5.7.

(f) $\Rightarrow$ (g): For f analytic nonvan on U, $\log|f|$ is harmonic. Let g be analytic with $Re(g) = \log|f|$. Then $|e^g| = |f|$ on U $\Rightarrow$ $|fe^{-g}| = 1$ on U hence fe^{-g} constant by Max Modulus Thm say with $fe^{-g} = a \in \mathbb{C} - \{0\}$. Choose a log b of a, $a = e^b$. Then $f = e^{g+b}$ so $g+b$ is an analytic log.

(g) $\Rightarrow$ (h): If f has analytic log h, then $e^{h/2}$ is an analytic square root of f

(h) $\Rightarrow$ (a): Wait for Riemann Mapping Thm! $\square$

Htpy for non-closed curves:

Thm If γ_0, γ_1 are homotopic paths in U, connecting z_0, z_1, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for all f analytic on U.

Thm U conn'd open, $z_0, z_1 \in U$. U is simply conn'd iff any two curves connecting z_0, z_1 are htpic in U.

Calculus of Residues

Thm If $f(z) = \dfrac{g(z)}{(z-z_0)^k}$ where g is analytic in a nbhd of z_0, then $\mathrm{Res}(f, z_0) = \dfrac{g^{(k-1)}(z_0)}{(k-1)!}$ is the coeff of $(z-z_0)^{k-1}$ in the power series for g at z_0.

Pf Clear. $\square$

e.g. If $f(z) = \dfrac{g(z)}{z-z_0}$ then $\mathrm{Res}(f, z_0) = g(z_0)$

If $f(z) = \dfrac{g(z)}{(z-z_0)^2}$ then $\mathrm{Res}(f, z_0) = g'(z_0)$.

e.g. $f(z) = \dfrac{z^2+1}{(z-1)(z^2-2z+5)} = \dfrac{g(z)}{z-1}$ for $g(z) = \dfrac{z^2+1}{z^2-2z+5}$ analytic at $z=1$. Thus $\mathrm{Res}(f, 1) = g(1) = \frac{1}{2}$.

e.g. $f(z) = \dfrac{\sin(z)}{z^2}$ then $\mathrm{Res}(f, 0) = \sin'(0) = \cos(0) = 1$.

Residue of a Quotient

Thm Suppose $f(z) = \dfrac{p(z)}{h(z)}$ for p, h analytic at z_0, h has a zero of order k at z_0. If we write $h(z) = (z-z_0)^k q(z)$ for q analytic at z_0 w/ $q(z_0) \neq 0$, then $\mathrm{Res}(f, z_0) = c_{k-1}$ for c_{k-1} the coeff of $(z-z_0)^{k-1}$ in the power series exp'n of $g(z) = \dfrac{p(z)}{q(z)}$.

Note May compute c_{k-1} by long division of power series.

Suppose $k=1$. Then $\mathrm{Res}(f, z_0) = c_0 = \dfrac{p(z_0)}{q(z_0)}$

Since $h(z) = (z-z_0)q(z)$, $q(z_0) = h'(z_0)$ and we get

Cor For p, h analytic at z_0 where h has a zero of order 1 at z_0,
$$\mathrm{Res}(p/h, z_0) = p(z_0)/h'(z_0). \quad \square$$

e.g. $\operatorname{Res}\left(1/\sin z, 0\right) = \dfrac{1}{\cos(0)} = 1.$

e.g. $f(z) = \dfrac{1}{e^z - 1 - z} = \dfrac{1}{z^2 q(z)}$ for $q(z) = \dfrac{1}{2} + \dfrac{z}{3!} + \cdots$.

To find $\operatorname{Res}(f,0)$, seek linear coeff of $1/q$:

$$
\begin{array}{r|l}
 & 2 - \frac{4}{6}z + \cdots \\
\left(\frac{1}{2} + \frac{z}{3!} + \cdots\right) & 1 \\
 & 1 + \frac{2z}{3!} + \cdots \\
\hline
 & -\frac{2z}{3!} + \cdots \\
 & +\frac{4z}{6} + \cdots \\
\hline
 & ?z^2 + \cdots
\end{array}
\implies \operatorname{Res}(f,0) = \frac{-2}{3}.
$$

e.g. $\operatorname{Res}\left(\dfrac{\cot z}{z^2}, 0\right) = ?$

$\dfrac{\cot z}{z^2}$ has a pole of order 3 at 0. Set $p(z) = \cos z$, $h(z) = z^2 \sin z$.

$$q(z) = \frac{h(z)}{z^3} = \frac{\sin z}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots$$

To compute c_2 for p/q:

$$
\begin{array}{r|l}
 & 1 - \frac{z^2}{3} + \cdots \\
1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots & 1 - \frac{z^2}{2} + \frac{z^4}{12} - \cdots \\
 & 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots \\
\hline
 & -\frac{z^2}{3} + \frac{9z^4}{120} - \cdots \\
 & -\frac{z^2}{3} + \frac{z^4}{18} + \cdots \\
\hline
 & ?z^4
\end{array}
\implies \operatorname{Res}\left(\frac{\cot z}{z^2}, 0\right) = -\frac{1}{3}.
$$

<u>Evaluating</u> <u>integrals</u> <u>using</u> <u>residues</u>

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

so $\cos\theta = \left[\frac{z+z^{-1}}{2}\right](e^{i\theta})$, $\sin\theta = \left[\frac{z-z^{-1}}{2i}\right](e^{i\theta})$.

Thus if f is a rational expression in $\cos\theta$, $\sin\theta$ we may evaluate $\int_0^{2\pi} f(\theta)\,d\theta$ by letting $g(z)$ be the rat'l fn in z where $\cos\theta$ replaced by $\frac{z+z^{-1}}{2}$, $\sin\theta$ by $\frac{z-z^{-1}}{2i}$ so that $g(e^{i\theta}) = f(\theta)$ and

$$\int_\gamma \frac{g(z)}{iz}\,dz \quad \text{for } \gamma(\theta) = e^{i\theta} \text{ on } [0,2\pi].$$
$$= \int_0^{2\pi} f(\theta)\,d\theta.$$

<u>e.g.</u> $f(\theta) = \frac{1}{2+\sin\theta} \rightsquigarrow \frac{1}{2+(z-z^{-1})/2i} = \frac{2iz}{4iz+z^2-1} =: g(z)$

Dividing by iz, want to compute $\int_{|z|=1} \frac{2}{z^2+4iz-1}\,dz$

$= \int_{|z|=1} \frac{2}{(z+(2-\sqrt{3})i)(z+(2+\sqrt{3})i)}\,dz$ Only pole in $D_1(0)$ is $(-2+\sqrt{3})i$. The residue here is $\frac{2}{(-2+\sqrt{3})i+(2+\sqrt{3})i} = \frac{-\sqrt{3}}{3}i$

By the residue thm,

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = 2\pi i \cdot \left(\frac{-\sqrt{3}}{3}i\right)\cdot 1$$
$$= \frac{2\sqrt{3}}{3}\pi.$$

Improper Integrals

call $\lim_{x\to\infty}\int_{-x}^{x} f(t)\,dt$ the principal value of $\int_{-\infty}^{\infty} f(t)\,dt$, denoted

$\text{P.V.}\int_{-\infty}^{\infty} f(t)\,dt$. If $\int_{\mathbb{R}} f$ exists $\left(= \lim_{x,y\to\infty}\int_{-x}^{y} f(t)\,dt\right)$ then

$\text{PV}\int_{\mathbb{R}} f = \int_{\mathbb{R}} f$.

Suppose $U \subseteq \mathbb{C}$ open, $H \subseteq U$ for $H = \{z \mid \operatorname{Im} z \geq 0\}$ or $\{z \mid \operatorname{Im} z \leq 0\}$.

f meromorphic on U with no sings on $\mathbb{R}$.

Suppose $\exists\, p>1$, $R, C>0$ s.t. $|f(z)| \underset{\circledast}{\leq} C|z|^{-p}$ for $z\in H$, $|z|>R$.

Let γ_r be  or 

Then $\int_{\gamma_r} f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_0^{\pi} f(re^{it})\, ire^{it}\,dt$ (in upper case)

If $r>R$, then $\circledast$ gives $|f(re^{it})\,ire^{it}| \leq Cr^{1-p}$ and so

$$\left|\int_0^{\pi} f(re^{it})\,ire^{it}\,dt\right| \leq \pi C r^{1-p} \xrightarrow[r\to\infty]{} 0 \quad \text{since } p>1.$$

Thus $\lim_{r\to\infty}\int_{\gamma_r} f(z)\,dz = \text{P.V.}\int_{-\infty}^{\infty} f(t)\,dt$ (in fact, $\int_{\mathbb{R}} f$ converges under $\circledast$).

By the residue theorem, $\int_{\gamma_r} f(z)\,dz = 2\pi i \sum_{\substack{z \text{ inside} \\ \gamma_r,\ \text{sing of } f}} \operatorname{Res}(f,z)$

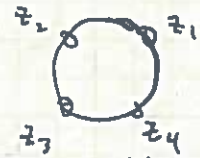
By $\circledast$, no sings outside $|z|=R$, so only finitely many sings in upper half plane $\Longrightarrow$ integral ind of r for $r>R$ and $= 2\pi i \sum_{\substack{z\in H \\ \text{sing}}} \operatorname{Res}(f,z)$ (for lower H, $-2\pi i\sum \operatorname{Res}$)

<u>e.g.</u> $\displaystyle\int_{\mathbb{R}} \frac{x^2}{1+x^4}\,dx = ?$

Set $f(z) = \dfrac{z^2}{1+z^4} = \dfrac{z^{-2}}{z^{-4}+1}$. Set $R>1$. Get

$$|f(z)| \le \frac{|z|^{-2}}{1-R^{-4}} \quad \text{for } |z|\ge R \text{ so } * \text{ satisfied w/ } p=2,\ C=\frac{1}{1-R^{-4}}$$

The poles of f are at 4^{th} roots of -1:



$f(z) = \dfrac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$ ↳ eval at z_j w/ $z-z_j$ removed to get $\operatorname{Res}(f,z_j)$ $j=1,2$

$\operatorname{Res}(f,z_1) = -\dfrac{\sqrt{2}}{8}(1+i)$

$\operatorname{Res}(f,z_2) = \dfrac{\sqrt{2}}{8}(1-i)$

$\Sigma \operatorname{Res} = -\dfrac{\sqrt{2}}{4} i \implies \displaystyle\int_{\mathbb{R}} \frac{x^2}{1+x^4}\,dx = \frac{\sqrt{2}}{2}\pi.$

<u>Read</u> eg. 5.2.5, 5.2.6.