Laurent Series

Defn. A neighborhood of ∞ is any open set in $\mathbb{C}$ containing the complement of a closed bdd disc.

- If f is analytic in a nbhd of ∞, say it vanishes at ∞ if $\lim_{z\to\infty} f(z) = 0$.

Lemma If h is analytic on $\mathbb{C} - \overline{D}_r(z_0)$ and vanishes at ∞, then

$$q(w) = \begin{cases} h(1/w + z_0) & \text{if } w \neq 0 \\ 0 & \text{if } w = 0 \end{cases}$$

is analytic on $D_{1/r}(0)$.

Pf $\frac{1}{w} + z_0 \in \mathbb{C} - \overline{D}_r(z_0)$ iff $\frac{1}{|w|} > r$ iff $w \in D_{1/r}(0)$.

Clearly q is analytic on $D_{1/r}(0) - \{0\}$ and $\lim_{w\to 0} q(w) = 0$ so q is analytic on $D_{1/r}(0)$. $\square$

Defn An open annulus centered at z_0 is a set of the form

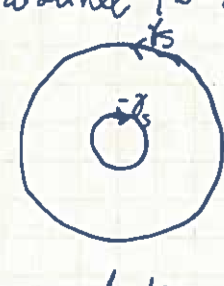
$$A = \{ z \in \mathbb{C} \mid r < |z - z_0| < R \}$$

where $0 \leq r < R \leq \infty$.

For $r < s < S < R$ consider γ_s, γ_S in A pos around $|w - z_0| = s$, $|w - z_0| = S$.

Define $\Gamma = \gamma_S - \gamma_s$. Then

$$Ind_\Gamma(z) = \begin{cases} 0 & \text{if } |z - z_0| > S \\ 1 & \text{if } s < |z - z_0| < S \\ 0 & \text{if } |z - z_0| < s. \end{cases}$$



Hence Γ nullhomologous in A, so if f is analytic on A, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w - z}\,dw = \begin{cases} 0 \\ f(z) \\ 0 \end{cases}$$

Thus if s, S are on the same side of $|z-z_0|$,

$$\frac{1}{2\pi i}\int_{\gamma_S} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w-z}\,dw.$$

If s, S are on opp sides of $|z-z_0|$,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_S} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w-z}\,dw.$$

<u>Thm</u> If f is analytic on an annulus A then $\exists!$ way to write

$$f(z) = g(z) - h(z) \quad \text{for } z \in A$$

where g is analytic on $D_R(z_0)$, and h is analytic on $\mathbb{C} \setminus \overline{D}_r(z_0)$ and vanishes at ∞.

<u>Pf</u> Define g as follows: if $|z-z_0| < R$, choose S with $|z-z_0| < S$ and $r < S < R$. Set $g(z) = \frac{1}{2\pi i}\int_{\gamma_S} \frac{f(w)}{w-z}\,dw$. By above, this doesn't depend on choice of S with $|z-z_0| < S < R$.

Define h on $\mathbb{C} \setminus \overline{D}_r(z_0)$ as follows: for $|z-z_0| > r$, choose s s.t. $r < s < R$, $s < |z-z_0|$. Set

$$h(z) = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w-z}\,dw$$

which doesn't depend on s

If $z \in A$, $f(z) = g(z) - h(z)$ by prior work. Use Morera to get that g, h are analytic in appropriate domains

- $h(z) \to 0$ as $z \to \infty$: simple check.
- uniqueness : Identity Thm $\square$

<u>Laurent Series Expansion</u>

<u>Thm</u> If f is analytic on annulus A, then f has a unique exp'n of the form $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ which converges to f at all pts of A and converges unif on cpt subsets of A.

Pf Write $f = g - h$ as in previous thm. Then

$$g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \quad \text{on } D_R(0)$$

and $q(w)$ as in Lemma is analytic on $D_{1/r}(0)$ with $q(0)=0$

$$\Rightarrow \quad q(w) = \sum_{n=1}^{\infty} b_n w^n = h(1/w + z_0)$$

Subbing $z = \frac{1}{w} + z_0$ (i.e. $w = (z-z_0)^{-1}$) get

$$h(z) = \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

Set $c_n = -b_{-n}$ for $n<0$ to get $h(z) = -\sum_{n=-\infty}^{-1} c_n (z-z_0)^n$.

Then $f = g - h$ on A give $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$. $\square$

e.g. $f(z) = \dfrac{1}{(z-1)(z-2)}$ on $A = \{1 < |z| < 2\}$

If $g(z) = \frac{1}{z-2}$ for $|z|<2$ and $h(z) = \frac{1}{z-1}$ for $|z|>1$, then

$f = g - h$ on A and $h(z) \to 0$ at ∞.

$$g(z) = \frac{-1/2}{1 - z/2} = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} \quad \text{in } D_2(0)$$

$$h(z) = \frac{1}{z-1} = \frac{1/z}{1 - 1/z} = \sum_{n=1}^{\infty} \frac{1}{z^n} \quad \text{on } \mathbb{C} \setminus \overline{D}_1(0). \quad \text{Thus}$$

$$f(z) = \sum_{n=-\infty}^{-1} (-1) z^n + \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} z^n \quad \text{on } A.$$

Read On $D_1(0) \setminus \{0\}$, $f(z) = -\sum_{n=-1}^{\infty} (z-1)^n$.

e.g. $f(z) = e^{1/z}$ analytic on $\mathbb{C} \setminus \{0\}$, $f(z) \to 1$ as $z \to \infty$ so

$g(z) = 1$, $h(z) = 1 - e^{1/z}$, $f(z) = \sum_{n=-\infty}^{0} \frac{z^n}{|n|!}$

Thm [Integral fmla for Laurent series coeffs] If $A = \{r < |z - z_0| < R\}$, f analytic on A, $r<s<R$, then the Laurent series of f on A has coeffs

$$c_k = \frac{1}{2\pi i}\int_{|w-z_0|=s} \frac{f(w)}{(w-z_0)^{k+1}}\, dw.$$

Pf We have
$$\frac{1}{2\pi i}\int_{|w-z_0|=s} \frac{f(w)}{(w-z_0)^{k+1}}\,dw = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{|w-z_0|=s} (w-z_0)^{n-k-1}\,dw}_{\int_0^{2\pi} i s^{n-k} e^{i(n-k)t}\,dt}$$

$$= \begin{cases} 0 & \text{if } n \neq k \\ 2\pi i & \text{if } n = k \end{cases}. \qquad \square$$

The Residue Thm

$D_r(z_0) \setminus \{z_0\}$ is an annulus so when f is analytic on $D_r(z_0)\setminus\{z_0\}$ with isolated singularity at z_0, then $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$

Defn The coefficient of $(z-z_0)^{-1}$ is the residue of f at z_0.

$$\operatorname{Res}(f, z_0) := c_{-1}.$$

e.g. $\operatorname{Res}\left(\frac{g(z)}{z-z_0}, z_0\right) = g(z_0)$ for g analytic on open $\ni z_0$.

Thm For f analytic on $D_R(z_0)\setminus\{z_0\}$ with isolated sing at z_0 and $0<r<R$,

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)\,dz. \qquad \square$$

Residue Thm Let f be analytic on $U\setminus E$, $U \subseteq \mathbb{C}$ open, E discrete subset of U. If γ is a closed path in $U\setminus E$ which is homologous to 0 in U, then

(a) there are only finitely many pts of E at which $\operatorname{Ind}_\gamma$ is nonzero

(b) if these pts are $\{z_1, \dots, z_n\} \subseteq E$, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \operatorname{Ind}_\gamma(z_j)\operatorname{Res}(f, z_j).$$

Pf (a) Choose $r>0$ s.t. $\gamma(I) \subseteq D_r(0)$. Then the bdd cpts of $\mathbb{C}\setminus\gamma(I)$ are inside $D_r(0)$. $\operatorname{Ind}_\gamma$ is nonzero only on (some) bdd cpts of $\mathbb{C}\setminus\gamma(I)$. Thus $\gamma(I) \cup \{\text{cpts of } \mathbb{C}\setminus\gamma(I) \text{ on which } \operatorname{Ind}_\gamma \text{ is nonzero}\}$ is a bdd set K. $K = \mathbb{C} \setminus \{\text{cpts of } \mathbb{C}\setminus\gamma(I) \text{ on which } \operatorname{Ind}_\gamma = 0\}$ so K is closed. Thus K is compact.

Since γ nullhom in U, $K \subseteq U$. Choose for each point of U an open disc containing either no sings, or only one sing at its center. (Possible since E discrete.) This is an open cover of K hence it contains a finite subcover. $\Rightarrow$ only finitely many sings of f in K. ✓(a)

(b) Let $z_1, \dots, z_n$ be the sings of f at which $\mathrm{Ind}_\gamma \neq 0$. For each z_j choose $r_j > 0$ s.t. $D_{r_j}(z_j) \subseteq U - \gamma(I)$. Choose $0 < r < \min\{r_1, \dots, r_n\}$ s.t. $\bigcap_{j=1}^{n} \bar{D}_r(z_j) = \emptyset$. Then set $m_j = \mathrm{Ind}_\gamma(z_j)$ and define 1-cycle $\Gamma = \gamma - \sum_{j=1}^{n} m_j \gamma_j$ with $\gamma_j(t) = z_j + re^{2\pi i t}$ for $t \in [0,1]$.

Have each $\bar{D}_r(z_j) \subseteq K \subseteq U \Rightarrow \mathbb{C} - U \subseteq \bar{D}_r(z_j)$

$\Rightarrow \mathrm{Ind}_{\gamma_j}(z) = 0$ on $z \in \mathbb{C} - U$, and same for γ.

Thus Γ is nullhomologous in U.

Also have $\mathrm{Ind}_\gamma(z_j) = m_j = m_j \, \mathrm{Ind}_{\gamma_j}(z_j)$

and $m_j \, \mathrm{Ind}_{\gamma_j}(z_k) = 0$ for $k \neq j$.

Thus Γ is also nullhomologous in $U - E$ where f is analytic.

By the general Cauchy integral fmla,

$$0 = \int_\Gamma f(z)\,dz = \int_\gamma f(z)\,dz - \sum_{j=1}^{n} \underbrace{m_j}_{\mathrm{Ind}_\gamma(z_j)} \underbrace{\int_{\gamma_j} f(z)\,dz}_{\mathrm{Res}(f, z_j)}.$$

$\square$

e.g. Let γ be a simple closed path with 1, 2 inside $\gamma(I)$.

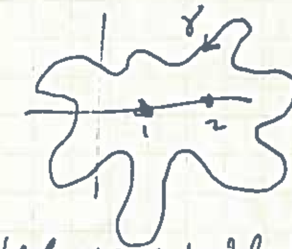
We determine $\displaystyle\int_\gamma \underbrace{\frac{z+1}{(z-1)(z-2)}}_{f(z)}\,dz = 2\pi i\left(\mathrm{Res}(f,1) + \mathrm{Res}(f,2)\right)$.

Let $g(z) = \frac{z+1}{z-2}$ so that g is analytic at 1 and $f(z) = \frac{g(z)}{z-1}$.

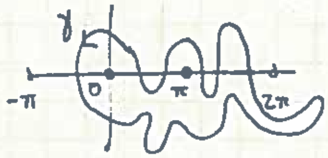
Thus $\mathrm{Res}(f,1) = g(1) = -2$. Working similarly with

$h(z) = \frac{z+1}{z-1}$, get $\mathrm{Res}(f,2) = h(2) = 3$. Thus

$$\int_\gamma \frac{z+1}{(z-1)(z-2)}\,dz = 2\pi i\,(3-2) = 2\pi i$$



e.g.



$f(z) = \frac{1}{\sin z}$ has isolated sings inside γ at 0 and π. The fn $g(z) = \frac{z}{\sin(z)}$ has a removable sing at 0 and the value of the corr analytic fn at 0 is 1. Thus $\operatorname{Res}(f,0) = 1$. Since $\sin(z) = -\sin(z-\pi)$,

$h(z) = \frac{z-\pi}{\sin(z)} = -\frac{z-\pi}{\sin(z-\pi)}$ has a removable sing at $z=\pi$ and corr value at π is -1. Thus $\operatorname{Res}(f,\pi) = -1$. By the Residue Thm,

$$\int_\gamma \frac{dz}{\sin z} = 0.$$

<u>Counting zeroes and poles</u>

Recall: f <u>meromorphic</u> on U means it is analytic on $U \setminus E$ where $E \subseteq U$ is discrete and f has poles on E.

<u>Thm</u> If f is meromorphic on U and $z_0 \in U$, then

$$\operatorname{Res}(f'/f, z_0) = k,$$

where $k = \begin{cases} \text{order of the zero of } f \text{ at } z_0 \\ -(\text{order of the pole of } f \text{ at } z_0) \\ 0 \quad \text{no zero or pole at } z_0 \end{cases}$.

<u>Pf</u> May factor $f(z) = (z-z_0)^k g(z)$ where g is meromorphic on U with no zero or pole at z_0. Then

$$f'(z) = k(z-z_0)^{k-1} g(z) + (z-z_0)^k g'(z)$$

so

$$\frac{f'(z)}{f(z)} = \frac{k}{z-z_0} + \frac{g'(z)}{g(z)}$$

Since g'/g is analytic at z_0, $\operatorname{Res}(f'/f, z_0) = k$. $\square$

Combined with the residue thm, we get:

Thm Let f be meromorphic on $U \subseteq \mathbb{C}$ open and let γ be a closed path in U homologous to 0 in U. Assume no zeroes or poles of f on $\gamma(I)$, and suppose the zeroes and poles of f at which $\mathrm{Ind}_\gamma \neq 0$ are $z_1, \dots, z_n$. Set $k_j = \begin{cases} \text{order of the zero of } f \text{ at } z_0 \\ -(\text{order of the pole of } f \text{ at } z_0) \end{cases}$ and $m_j = \mathrm{Ind}_\gamma(z_j)$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^n m_j k_j \quad . \quad \square$$

Cor For $U, f, z_1, \dots, z_n, k_1, \dots, k_n, \gamma, m_1, \dots, m_n$'s in Thm, if we create the new path $f \circ \gamma$ then $\sum_{j=1}^n m_j k_j = \mathrm{Ind}_{f\circ\gamma}(0)$.

Pf If $\gamma : [a,b] \to U$ then

$$\mathrm{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z}\,dz = \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t)}{f\circ\gamma(t)}\,dt$$

$$= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\,\gamma'(t)}{f(\gamma(t))}\,dt = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

$$= \sum m_j k_j \quad . \quad \square$$

Cor If γ is a simple closed path, we get the above formulae with $m_j = 1$.

Application Suppose f is meromorphic in a convex open $U \subseteq \mathbb{C}$. Suppose the only zero of f in U is at z_1 of order k, and the only pole of f in U is at z_2, also of order k. Then $\exists$ analytic $g : U - [z_1, z_2] \to \mathbb{C}$ s.t. $f(z) = e^{g(z)}$ for $z \in U - [z_1, z_2]$. (Call g a logarithm of f.)

Pf Set $V = U \setminus [z_1, z_2]$. If γ is a closed path in V, then γ is homologous to 0 in U since U is convex. Note $\mathrm{Ind}_\gamma(z_1) = \mathrm{Ind}_\gamma(z_2)$ since z_1, z_2 are connected by a line segment in $\mathbb{C} \setminus V$ hence are in the same connected component of $\mathbb{C} \setminus \gamma(I)$. Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \mathrm{Ind}_\gamma(z_1)k - \mathrm{Ind}_\gamma(z_2)k = 0. \quad ⊛$$

Take $z_0 \in V$ fixed and z varying in V. Let γ_z be a path in V beginning at z_0, ending at z. Then

$$h(z) = \int_{\gamma_z} f'/f$$

is independent of the choice of path γ_z by ⊛.
Furthermore h is a primitive of f'/f, i.e. $h' = f'/f$, whence

$$\begin{aligned}(fe^{-h})' &= f'e^{-h} - fh'e^{-h}\\ &= f'e^{-h} - f'e^{-h}\\ &= 0.\end{aligned}$$

Thus $fe^{-h} = c$, constant, i.e. $f = ce^h$.
Set $g(z) = h(z) + \log(c)$ to get $e^{g(z)} = e^{h(z)}f(z)e^{-h(z)} = f(z)$ where $\log$ is any branch of the logarithm. □