Maximum Modulus Principle

<u>Thm</u> If f is analytic on a conn'd open set $U \subseteq \mathbb{C}$ and $|f|$ has a local max at $z_0 \in U$, then f is constant on U.

<u>Lemma</u> Let $f: I = [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ be cts. If

$$|f(t)| \leq M := \left| \frac{1}{b-a} \int_a^b f(t)\,dt \right|, \quad \forall t \in I$$

then f has constant modulus M on I.

<u>Pf Lemma</u> Choose $u \in \mathbb{C}$, $|u| = 1$ s.t. $u \int_a^b f(t)\,dt = \left| \int_a^b f \right|$.

Then $\int_a^b (M - uf(t))\,dt = 0$. Let $uf = g + ih$, $g, h: I \to \mathbb{R}$.

Then $|f(t)| \leq M \Rightarrow g(t) \leq M \Rightarrow M - g(t) \geq 0$.

Have $\int_a^b (M - g(t))\,dt = 0$ and $\int_a^x (M - g(t))\,dt$ diff'l in x w/ derivative $M - g(x) \geq 0 \Rightarrow$ non-decreasing fn.

Since 0 at $x = a$, $x = b$, must be constant $\Rightarrow M = g(t)$.

Thus $uf = M + ih$ and

$$M^2 \geq |f(t)|^2 = |uf(t)|^2 = g(t)^2 + h(t)^2 = M^2 + h(t)^2$$

$\Rightarrow h(t) = 0 \ \forall t \in I$. Thus $f = u^{-1} M$ which has modulus M. $\square$

<u>Pf Thm</u> Choose $r > 0$ s.t. $\overline{D}_r(z_0) \subseteq U$ and $|f(z_0)|$ max for $|f(z)|$ on $\overline{D}_R(z_0)$. By Cauchy's integral thm,

$$f(z_0) = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,dt.$$

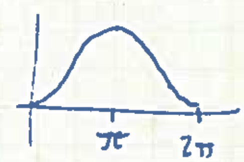
Since $|f(z_0 + re^{it})| \leq |f(z_0)|$ on $[0, 2\pi]$, may apply the lemma with $M = |f(z_0)|$. It follows that f is constant on $z_0 + re^{it}$, a non-discrete subset of U. By the identity thm, f is constant on all of U. $\square$

<u>Cor</u> Suppose U conn'd, bdd, open $\subseteq \mathbb{C}$. If f is cts on $\overline{U}$, analytic on U, and nonconstant, then $\max_{\overline{U}} |f(z)|$ is attained on ∂U and nowhere else.

<u>Pf</u> $|f|$ attains a max by EVT applied to $\overline{U}$. By the Thm, $|f|$ has no local max on U, so must on $\overline{U} - U = \partial U$. $\square$

<u>e.g.</u> Where does $f(z) = z^2 - z$ attain max modulus on $\overline{D}_1(0)$?
By Cor, on $S^1 = \{e^{it} \mid t \in [0, 2\pi]\}$, so only must maximize

$$h(t) = |e^{2it} - e^{it}|^2 = |e^{it} - 1|^2 = 2 - 2\cos t.$$



This is clearly at $t = \pi$, so max modulus of f is $|f(-1)| = 2$.

<u>Schwarz's Lemma</u> Let f be analytic on $D_1(0)$ w/ $f(0) = 0$ and $|f(z)| \leq 1$ for every $z \in D_1(0)$. Then $|f(z)| \leq |z|$ for all $z \in D_1(0)$ and $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then $f(z) = cz$ for some constant $c \in \mathbb{C}$.

<u>Pf</u> Since $f(0) = 0$, $f(z) = z\,g(z)$ with g analytic on $D_1(0)$.
Since $|f(z)| \leq 1$, $|g(z)| \leq \frac{1}{r}$ on $|z| = r$, for each $r < 1$.
By Max Modulus Thm, this also holds for $|z| < r$. Thus $|g(z)| \leq 1$ on $D_1(0)$. Hence $|f(z)| = |z||g(z)| \leq |z|$.
Now $f'(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} g(z) = g(0)$, so $|f'(0)| \leq 1$.
If $|f'(0)| = 1$, then $|g(0)| = 1$ is max modulus of g on $D_1(0)$
$\Rightarrow g$ constant. $\square$

Defn Let $U, V \subseteq \mathbb{C}$ open. A bi-analytic map from U to V is an analytic fn $f: U \to V$ with an analytic inverse $f^{-1}: V \to U$.

Thm The only bi-analytic maps $D_1(0) \to D_1(0)$ that take 0 to 0 are of the form $f(z) = cz$ for $|c| = 1$. I.e., just rotations.

Pf Both f, f^{-1} satisfy Schwarz's lemma, so $|f'(0)| \le 1$ and $|(f^{-1})'(0)| \le 1$. Applying the chain rule to $f^{-1} \circ f = id$, we have $(f^{-1})'(0) = \frac{1}{f'(0)} \implies |f'(0)| = 1$, and the conclusion follows from SL. $\square$

Harmonic Functions

Thm Let u be a function of class C^2 ~~and~~ harmonic on a convex open set U. Then u has a harmonic conjugate on U.

Pf Let $g = u_x - iu_y$, which is C^1 and

$$u_{xx} = -u_{yy}, \quad u_{xy} = u_{yx}$$

so g is analytic on U. Since U is convex, g has an ~~antideriv~~ primitive h on U, h analytic w/ $h' = g$. If $h = w + iv$, then $u_x - iu_y = g = h' = w_x + iv_x = w_x - iw_y$

$$\implies u_x = w_x, \quad u_y = w_y.$$

Thus $w = u + c$, $c \in \mathbb{R}$ $\implies f = h - c = u + iv$ analytic w/ $\mathrm{Re}(f) = u$. $\square$

Thm If u is harmonic on conn'd open U and u has a local max at some $z_0 \in U$, then u is constant on U.

Pf p. 105 $\square$

<u>Thm</u> If u is harmonic on $U \subseteq \mathbb{C}$ open, $\overline{D}_r(z_0) \subseteq U$, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt.$$

Pf Cauchy integral thm. $\square$

Chains and Cycles

Defn For $U \subseteq \mathbb{C}$ open, a 1-chain on U is a formal $\mathbb{Z}$-linear combination of paths $\gamma_i : [0,1] \to U$,

$$\Gamma = \sum_{j=1}^{p} m_i \gamma_i$$

where γ_i are distinct and $0 \cdot \gamma = 0$. These form an Abelian group under addition:

$$\sum_{i=1}^{p} m_i \gamma_i + \sum_{i=1}^{p} n_i \gamma_i = \sum_{i=1}^{p} (m_i + n_i) \gamma_i .$$

Note · Paths are 1-chains

· Every path is a sum of smooth paths

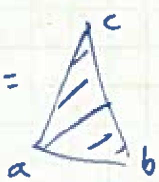
Defn For $\Gamma = \sum_{i=1}^{p} m_i \gamma_i$, $I = [0,1]$, set $\Gamma(I) = \bigcup_{m_i \neq 0} \gamma_i(I)$.

If f is cts on $E \subseteq \mathbb{C}$ with $\Gamma(I) \subseteq E$, define

$$\int_\Gamma f = \sum_{i=1}^{p} m_i \int_{\gamma_i} f .$$

Prop $\int_{\Gamma + \Lambda} f = \int_\Gamma f + \int_\Lambda f$, $\int_{m\Gamma} f = m \int_\Gamma f$ $\forall m \in \mathbb{Z}$. $\square$

Defn Suppose Γ, Λ are 1-chains with $\Gamma(I), \Lambda(I) \subseteq E \subseteq \mathbb{C}$. Call Γ, Λ E-equivalent if $\int_\Gamma f = \int_\Lambda f$ $\forall$ cts f on E.

e.g. $\Delta =$ 

then $\partial\Delta$ & $[a,b]+[b,c]+[c,a]$ & $[a,b]+[b,c]-[a$ are equivalent.

Defn A 0-chain in U is a $\mathbb{Z}$-linear combination of singleton subsets of $\mathbb{C}$, $\sum_{i=1}^{p} m_i \{z_i\}$, $m_i \in \mathbb{Z}$, $z_i \in \mathbb{C}$.

$$\partial\left(\sum_{i=1}^{p} m_i \gamma_i\right) = \sum_{i=1}^{p} \left(m_i \{\gamma_i(1)\} - m_i \{\gamma_i(0)\}\right)$$

(combine any like terms)

Note $\partial(\Gamma+\Lambda) = \partial(\Gamma) + \partial(\Lambda)$ so ∂ is a group homomorphism (it's $\mathbb{Z}$-linear)

Defn A 1-chain Γ in U is a *cycle* if $\partial\Gamma = 0$.

Thm If Γ is a 1-cycle, then there is a 1-cycle Λ equivalent to Γ which is a sum of closed paths.

Pf We make changes to Γ which don't change integrals over it or $\Gamma(I)$: First write Γ as a sum of paths w/ coeff 1:

$$m\gamma \to \gamma + \gamma + \cdots + \gamma \quad \text{if } m>0$$
$$\to (-\gamma) + (-\gamma) + \cdots + (-\gamma) \quad \text{if } m<0$$

This results in $\tilde{\Gamma}$, a sum of n paths. If not all paths closed, have γ_j in $\tilde{\Gamma}$ with $\gamma_j(0) \neq \gamma_j(1)$. Since $\partial\tilde{\Gamma} = 0$, know $\gamma_j(1) = \gamma_k(0)$ for some term γ_k of $\tilde{\Gamma}$. Join γ_j and γ_k to express $\tilde{\Gamma}$ as $\tilde{\tilde{\Gamma}}$ with $n-1$ terms.

Proceeding by induction, get a sum of closed paths. $\square$

Index of a cycle

Defn If Γ is a 1-cycle and $z \in \mathbb{C} \setminus \Gamma(I)$, define

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w-z}.$$

the *index* of Γ around z.

Thm Γ a 1-cycle in $\mathbb{C}$. Then

(a) $\mathrm{Ind}_\Gamma : \mathbb{C} \setminus \Gamma(I) \to \mathbb{Z}$

(b) Ind_Γ is locally constant

(c) Ind_Γ is 0 on the unbdd cpt of $\mathbb{C} \setminus \Gamma(I)$.

(d) If Λ is a cycle, $z \in \mathbb{C} \setminus (\Gamma(I) \cup \Lambda(I))$, then

$$\mathrm{Ind}_{\Gamma+\Lambda}(z) = \mathrm{Ind}_\Gamma(z) + \mathrm{Ind}_\Lambda(z).$$

e.g. $\gamma(t) = 2e^{2\pi i t}$, $\lambda(t) = e^{2\pi i t}$, $t \in [0,1]$ then $\mathrm{Ind}_{\gamma-\lambda}(z) = 0$ everywhere

Homologous Cycles

Defn $U \subseteq \mathbb{C}$ open, Γ, Λ 1-cycles in U are <u>homologous</u> in U if

$$\mathrm{Ind}_\Gamma(z) = \mathrm{Ind}_\Lambda(z)$$

for all z in $\underline{\mathbb{C} \setminus U}$. Call Γ <u>homologous to 0</u> in U if $\mathrm{Ind}_\Gamma(z) = 0 \ \forall z \in \mathbb{C} \setminus U$.

Intuition: Components of Γ don't "go around any holes in U"

U

homologous to 0

not homologous to 0.

Note Γ homologous to Λ iff $\Gamma - \Lambda$ homologous to 0.

Cauchy's Theorems

For f analytic on $U \subseteq \mathbb{C}$ open, define

$$g(z,w) = \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases},$$

a well-defined function $g : U \times U \to \mathbb{C}$.

Lemma g is continuous.

Pf Clearly cts for $w \neq z$. Need to show

$$\lim_{(z,w)\to(z_0,z_0)} g(z,w) = f'(z_0).$$

If $z \neq w$, $f(w)-f(z) = \int_z^w f'(\lambda)\,d\lambda$ so

$$|g(z,w) - f'(z_0)| = \Big| \frac{1}{w-z} \int_z^w \underbrace{(f'(\lambda) - f'(z_0))}_{\text{small by continuity of } f'} d\lambda \Big|$$

If $z = w$, then $|g(z,w) - f'(z_0)| = |f'(z) - f'(z_0)|$ is again small. $\square$

Cauchy's Integral Formula Let $U \subseteq \mathbb{C}$ open, f analytic on U, Γ a 1-cycle in U homologous to 0 in U. Then

$$\mathrm{Ind}_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\,dw \quad \forall z \in U - \Gamma(I).$$

Pf Let $h(z) = \int_\Gamma g(z,w)\,dw$, which is cts by Lemma

$$= \int_\Gamma \frac{f(w)}{w-z}\,dw - \int_\Gamma \frac{f(z)}{w-z}\,dw = \int_\Gamma \frac{f(w)}{w-z}\,dw - 2\pi i\, \mathrm{Ind}_\Gamma(z) f(z)$$

Thus need to show $h(z) \equiv 0$ on U. To do so, we prove h is entire, bounded, with $\lim_{z\to\infty} h(z) = 0$.

For Δ a triangle in U,

$$\int_{\partial\Delta} h(z)\,dz = \int_{\partial\Delta}\int_\Gamma g(z,w)\,dw\,dz$$

$$= \int_\Gamma \int_{\partial\Delta} \underbrace{g(z,w)\,dz}\,dw \qquad \text{[Fubini]}$$

for fixed w, analytic for $z \neq w$,
so Cauchy's Thm on $\partial\Delta$:

$$= \int_\Gamma 0\,dw = 0.$$

By Morera's Thm, h is analytic on U.
Let $V = \{z \in \mathbb{C} \setminus \Gamma(I) \mid \mathrm{Ind}_\Gamma(z) = 0\} \subseteq \mathbb{C}$ open.
Then $V \supseteq \mathbb{C} \setminus U$ by hypothesis, hence $U \cup V = \mathbb{C}$.
If $z \in V \cap U$, then $\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{w-z}\,dw = f(z)\,\mathrm{Ind}_\Gamma(z) = 0$.
Thus $h(z) = \int_\Gamma \frac{f(w)}{w-z}\,dw - \int_\Gamma \frac{f(z)}{w-z}\,dw = \int_\Gamma \frac{f(w)}{w-z}\,dw$ for $z \in U \cap V$.
Thus extend h to $\mathbb{C}$ by defining it to be $\int_\Gamma \frac{f(w)}{w-z}\,dw$ on V.
On the unbounded component of $\mathbb{C} \setminus \Gamma(I)$, h is given by

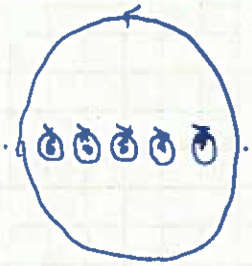
$$h(z) = \int_\Gamma \frac{f(w)}{w-z}\,dw \longrightarrow 0 \quad \text{as } z \to \infty.$$

By Liouville's Thm, $h \equiv 0$. $\square$

<u>Thm</u> [Cauchy] If f is analytic on $U \subseteq \mathbb{C}$ open and Γ is a 1-cycle in U homologous to 0 on U, then $\int_\Gamma f(z)\,dz = 0$.

<u>Pf</u> Fix $z_0 \in U \setminus \Gamma(I)$ and define $g(z) = f(z)(z - z_0)$. Then g is analytic on U with $g(z_0) = 0$. Thus $\int_\Gamma f(z)\,dz = \int_\Gamma \frac{g(z)}{z - z_0}\,dz$
$= 2\pi i\,\mathrm{Ind}_\Gamma(z_0)\,g(z_0) = 0$. $\square$

e.g. Γ



Γ is homologous to 0 in $\mathbb{C} \setminus \mathbb{Z}$

Thus $\displaystyle\int_\Gamma \frac{1}{\sin(\pi z)}\, dz = 0$.

Simple Closed Paths

Defn A closed curve $\gamma : [a,b] \to \mathbb{C}$ is simple if $\gamma(s) \neq \gamma(t)$ for $a \le s < t \le b$ unless $s = a$ and $t = b$.

A simple closed path is a simple closed curve which is piecewise smooth with non-zero left and right derivatives at each pt.

(Weak) Jordan Curve Thm If γ is a simple closed path, then $\mathbb{C} \setminus \gamma(I)$ has exactly two components: a bounded component on which $\mathrm{Ind}_\gamma(z) = \pm 1$, and an unbounded component on which $\mathrm{Ind}_\gamma(z) = 0$.

Pf Choose for each $z \in \gamma(I)$ an open disc D_z centered at z with $D_z \setminus \gamma(I)$ consisting of components L_z, R_z with $\mathrm{Ind}_\gamma(z)$ one unit greater on L_z than on R_z. Sufficiently close z have overlapping D_z with overlapping L's and overlapping R's.

$L := \bigcup_{z \in \gamma(I)} L_z$, $R := \bigcup_{z \in \gamma(I)} R_z$ connected open sets, and

$$\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_\gamma(w) + 1 \quad \text{for } z \in L,\ w \in R.$$

Thus $L \cap R = \emptyset$. Let $U := \bigcup D_z = L \cup R \cup \gamma(I)$

Every component of $\mathbb{C} \setminus \gamma(I)$ has a subset of $\gamma(I)$ as its boundary. Thus every such component has nonempty intersection with U, hence meets L or R. By connectedness, only meets 1. Thus there are only two cpts, one containing L, the other R. One is unbounded with $\mathrm{Ind}_\gamma = 0$ on it. $\square$

<u>Thm</u> If γ is a simple closed path and f is analytic on $U \subseteq \mathbb{C}$ open containing $\gamma(I)$ and its inside, then

$$\int_\gamma f(w)\,dw = 0$$

and
$$f(z) = \frac{\pm 1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

for each z on the inside of $\gamma(I)$. $\square$