Zeroes and Singularities

Thm If f is a function analytic on $U \subseteq \mathbb{C}$ open, then $\forall z_0 \in U$, exactly one of the following is true.

(a) there is an open disc $D_r(z_0)$ on which $f = 0$.

(b) there is a nonnegative integer k, open disc $D_r(z_0)$, and fn g, analytic on U, s.t.

$$f(z) = (z-z_0)^k g(z) \quad \forall z \in D_r(z_0)$$

and $g(z) \neq 0 \;\; \forall z \in D_r(z_0)$.

Pf Know $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ on $D_R(z_0)$ for some $R > 0$.

If all $c_n = 0$, then (a) holds. O/w call k smallest index s.t. $c_k \neq 0$. Then $f(z) = \sum_{n=k}^{\infty} c_n (z-z_0)^n = (z-z_0)^k \sum_{n=0}^{\infty} c_{n+k} (z-z_0)^n$.

Define $g(z) = \begin{cases} \sum_{n=0}^{\infty} c_{n+k} (z-z_0)^k & \text{if } z \in D_R(z_0) \\ \dfrac{f(z)}{(z-z_0)^k} & \text{if } z \in U \setminus \{z_0\}. \end{cases}$

Now $g(z_0) = c_k \neq 0$ and g is cts, so $g(z) \neq 0$ on some $D_r(z_0)$, $0 < r < R$.

This gives (b). $\square$

Q What is the distribution of zeroes of an analytic fn?

Thm Suppose f analytic on a connected open $U \subseteq \mathbb{C}$ and f is not identically 0. Then

(a) for each $z_0 \in U$, $\exists k \in \mathbb{N}$, $r > 0$, $g: U \to \mathbb{C}$ analytic s.t.

$$f(z) = (z-z_0)^k g(z) \quad \forall z \in U$$

and $g(z) \neq 0 \;\; \forall z \in D_r(z_0)$

(b) $\forall z_0 \in U \;\; \exists r > 0$ st f has no zeroes on $D_r(z_0)$ except possibly at z_0.

(c) The set of zeroes of f is at most countable.

Pf Let $V_1 = \{z_0 \in U \mid$ (a) from previous Thm holds $\}$

$V_2 = \{z_0 \in U \mid$ (b) from previous thm holds $\}$.

Then V_1, V_2 open, $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = U$. Since U conn'd, one of $V_j = \emptyset$, the other is U. But $V_1 = U$ contradicts $f \not\equiv 0$, thus $V_2 = U$, and this proves (a) & (b) of this thm.

To prove (c), modify discs of (b) s.t. centers are in $\mathbb{Q}(i)$, radii $\in \mathbb{Q}_{>0}$, z_0 still only 0 of f in the disc.

$|\mathbb{Q}^3| = \aleph_0$. $\square$

Note For $f: U \subseteq \mathbb{C} \to \mathbb{C}$ (U open), let $Z(f) = \{z \in U \mid f(z) = 0\}$.

Then $Z(f)$ consists of isolated points.

Defn For $E \subseteq U \subseteq \mathbb{C}$ (U open), call E a <u>discrete</u> subset of U if every point z_0 of U has a nbhd containing no points of E except possibly z_0 itself.

Then (b) of Thm says that for $f: U \subseteq \mathbb{C} \to \mathbb{C}$ analytic (U open conn'd), $Z(f)$ is a discrete subset of U.

discrete isolated, not discrete

Thm $f, g: U \to \mathbb{C}$ analytic on open conn'd $U \subseteq \mathbb{C}$. If $f(z) = g(z)$ $\forall z$ in a nondiscrete subset E of U, then $f \equiv g$.

Pf Let $h = f - g$. Then h analytic, $= 0$ on a nondiscrete subset of U, so $h \equiv 0$ on U. $\square$

e.g. $\cos z - 1 = \frac{-z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$

$$= \underset{\substack{\text{order 2}\\ \text{zero at}\\ 0}}{z^2} \underbrace{\left(-\frac{1}{2!} + \frac{z^2}{4!} - \cdots + (-1)^n \frac{z^{2n-2}}{(2n)!} + \cdots \right)}_{g(z)}$$

Thm (a) If an analytic fn g is not 0 at z_0 in its domain, then in some nbhd V of z_0 there is an analytic fn $h: V \to \mathbb{C}$ s.t. $g(z) = e^{h(z)}$. (b) If z_0 is a zero of order k for f, then $f(z) = (z-z_0)^k e^{h(z)}$ for some analytic h on nbhd V of z_0.

Pf (a) Choose a branch of log that does not have $g(z_0)$ on its cut line. Let W be its domain. Set $V = g^{-1}(W)$ and $h(z) = \log(g(z))$. Then $h(z) = \log(g(z))$. $\square$

Isolated Singularities

Defn If $U \subseteq \mathbb{C}$ open, $z_0 \in U$, f analytic on $U \setminus \{z_0\}$ but not on U, say f has an isolated singularity at z_0. If f can be given a value at z_0 so that it becomes analytic on U, call the singularity removable.

Thm If f has an isolated singularity at z_0 and is bounded in some deleted nbhd of z_0, then z_0 is a removable singularity of f.

Pf Suppose f is analytic and bounded on $U \setminus \{z_0\}$. Define g by $g(z) = (z-z_0)^2 f(z)$ for $z \neq z_0$, and $g(z_0) = 0$. Then

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z-z_0) f(z) = 0 \quad (\text{b/c } f \text{ is bounded})$$

Thus g is analytic on U. Since $g(z_0) = g'(z_0) = 0$, know that the first two terms in its power series about z_0 are 0. So factor out $(z-z_0)^2$ to get $g(z) = (z-z_0)^2 h(z)$ for h analytic, defined by power series at z_0. In a sufficiently small disc,

$$g(z) = (z-z_0)^2 f(z) = (z-z_0)^2 h(z) \implies f = h \text{ in this disc. } \square$$

<u>e.g.</u> $f(z) = \dfrac{\cos z - 1}{z^2}$ has a removable singularity at 0.

<u>Defn</u> A fn $f : U \setminus \{z_0\} \to \mathbb{C}$ of the form

$$f(z) = \frac{g(z)}{(z-z_0)^k}$$

with g analytic on U, $g(z_0) \neq 0$, $k \in \mathbb{Z}^+$, is said to have a <u>pole of order k</u> at z_0. If $k = 1$, call this a <u>simple pole</u>. An isolated singularity which is not removable and not a pole is called <u>essential</u>.

<u>e.g.</u> $f(z) = \dfrac{1}{1-e^z}$ has simple poles at $\{2\pi k i \mid k \in \mathbb{Z}\}$.

Indeed, $(1-e^z)' = -e^z \neq 0$.

<u>Thm</u> If f is analytic on $U \setminus \{z_0\}$ and has an essential singularity at z_0, then for every open disc D centered at z_0 and contained in U, $f(D \setminus \{z_0\})$ is dense in $\mathbb{C}$.

<u>Pf</u> Read p. 99. $\square$

<u>Moral</u> Essential singularities are wild! Cf. Big & little Picard

<u>Thm</u> Let f be an analytic fn with isolated singularity at z_0. Then (a) f has removable sing at z_0 iff $\lim_{z \to z_0} f(z) \in \mathbb{C}$.

(b) f has a pole at z_0 iff $\lim_{z \to z_0} f(z) = \infty$

(c) f has an essential sing at z_0 iff $\lim_{z \to z_0} f(z)$ DNE. $\square$

e.g. $f(z) = e^{1/z}$ has an essential sing at 0.

$$f\left(\frac{1}{2\pi n i}\right) = e^{2\pi n i} = 1 \quad \forall n \in \mathbb{Z}$$

$$f\left(\frac{1}{n}\right) = e^{n} \quad \forall n \in \mathbb{Z}.$$

Meromorphic Functions

Defn Let $U \subseteq \mathbb{C}$ open, $E \subseteq U$ discrete. If f is analytic on $U \setminus E$ and has a removable sing or pole at all points of E, then f is called meromorphic on U.

Thm If U is a connected open set and f is a meromorphic fn on U, $f \not\equiv 0$, then $1/f$ is meromorphic.

Pf Know $Z(f)$ = zeroes of f is discrete, and $P(f)$ = poles of f is discrete. Thus $E = Z(f) \cup P(f)$ is discrete. The fn $1/f$ is analytic on $U \setminus E$. For $z_0 \in E$ $\exists D = D_r(z_0)$ where f analytic on $D - \{z_0\}$, f has a zero or pole at z_0.

If f has a zero of order k at z_0, then $f(z) = (z-z_0)^k g(z)$ for g analytic on D, $g(z_0) \neq 0$. Thus $\frac{1}{f(z)} = \frac{1/g(z)}{(z-z_0)^k}$ has a pole of order k at z_0.

If f has pole of order k at z_0, then $1/f$ has a zero of order k at z_0. $\square$

Examples, Examples, Examples

Note that when f has a pole of order k at z_0, then

$$f(z) = \frac{g(z)}{(z-z_0)^k} \quad \text{for } g \text{ analytic in } D_r(z_0),\ g(z_0) \neq 0.$$

But then g has a power series expansion $g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ with $c_0 \neq 0$. Hence

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^{n-k} \quad \text{on } D_r(z_0) \setminus \{z_0\}.$$

This is a Laurent series expansion of f.

Generally write these as $\displaystyle\sum_{n=-N}^{\infty} a_n (z-z_0)^n$.

pole of order N if $a_{-N} \neq 0$.

e.g. Let's find a Laurent series for $\dfrac{z}{z^2+1}$ at $z_0 = i$.

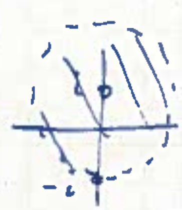
$$\frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)} = \frac{1}{2}\,\frac{1}{z-i} + \frac{1}{2}\,\frac{1}{z+i}.$$

$\dfrac{1}{z+i}$ analytic at i with power series

$$\frac{1}{z+i} = \frac{1}{2i + (z-i)} = \frac{1}{2i}\,\frac{1}{1 - \left(-\frac{z-i}{2i}\right)}$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{2i}\right)^n = \sum_{n=0}^{\infty} i^{-n-1} 2^{-n-1} (z-i)^n.$$

converging on $D_2(i)$



Thus $$\frac{z}{z^2+1} = \frac{1}{2}(z-i)^{-1} + \sum_{n=0}^{\infty} i^{n-1} 2^{-n-2} (z-i)^n.$$

e.g. What is the power series of $\frac{e^z}{1-z}$ at $z_0 = 0$?

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \text{conv on } \mathbb{C}$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \quad \text{conv on } D_1(0).$$

Hope
$$\frac{e^z}{1-z} = (1+z+z^2+z^3)\cdot\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$$

$$= 1 + (z+z) + \left(\frac{z^2}{2} + z^2 + z^2\right) + \left(\frac{z^3}{6} + \frac{z^3}{2} + z^3 + z^3\right) + \cdots$$

$$= 1 + 2z + \frac{5z^2}{2} + \frac{8z^3}{3} + \cdots \quad \text{for } |z| < 1.$$

Lemma Suppose $\sum a_n z^n$ and $\sum b_n z^n$ converge for $|z| < r_0$. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum c_n z^n$ converges for $|z| < r_0$ and

$$\sum c_n z^n = \left(\sum a_n z^n\right)\left(\sum b_n z^n\right).$$

Pf Let $f(z) = \sum a_n z^n$, $g(z) = \sum b_n z^n$. Both f, g analytic on $D_{r_0}(0)$, so $f \cdot g$ is analytic on $D_{r_0}(z_0)$. Thus

$$(fg)(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} z^n$$

Furthermore, $(fg)^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)$.

Hence
$$\frac{(fg)^{(n)}(0)}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} f^{(k)}(0)\, g^{(n-k)}(0)$$

$$= \sum_{k=0}^{n} a_k b_{n-k} = c_n. \quad \square$$

In particular, our Hope is true.

e.g. Find f s.t. $f(0)=0$ and $f'(x)=3x+2$.

Suppose f exists and is analytic, so $f(z)=\sum_{n=0}^{\infty} a_n z^n$.

Since $f'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$, we must have

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = 3\left(\sum_{n=0}^{\infty} a_n z^n\right)+2$$

i.e. $$\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n = (2+3a_0)+\sum_{n=1}^{\infty} 3a_n z^n$$

i.e. $$0=(2+3a_0-a_1)+\sum_{n=1}^{\infty}(3a_n-(n+1)a_{n+1})z^n.$$

Now $a_0=f(0)=0 \Rightarrow a_1=2$. For $n\geq 1$, $a_{n+1}=\frac{3a_n}{n+1}$

$\Rightarrow a_2=\frac{3a_1}{2}$, $a_3=\frac{3^2a_1}{3\cdot 2}$, $a_4=\frac{3^3a_1}{4\cdot 3\cdot 2}$, ...

and $a_n=\frac{3^{n-1}a_1}{n!}=3^n\left(\frac{2}{3}\right)n!$

Thus any power series sol'n is necessarily of the form

$$f(z)=\frac{2}{3}\sum_{n=1}^{\infty}\frac{3^n}{n!}z^n$$

$$=\frac{2}{3}(e^{3z}-1) \quad \checkmark$$

<u>TPS</u> Find Laurent series for

$$\frac{\cos z}{z^2}, \quad \frac{e^z-1}{z^2}, \quad \frac{z+1}{z-1}$$

at $z_0=0$.