Defn Let $\{f_n : S \subseteq \mathbb{C} \to \mathbb{C}\}$ be a sequence of functions.

Then (a) $\{f_n\}$ converges pointwise to the function $f : S \to \mathbb{C}$ if $\forall z \in S$, $\{f_n(z)\} \longrightarrow f(z)$.

(b) $\{f_n\}$ converges uniformly on S if $\forall \varepsilon > 0\ \exists N$ s.t.
$$|f_n(z) - f(z)| < \varepsilon \quad \forall n \geq N, z \in S.$$

Note In (a), N can depend on z; in (b) N is independent of z.

Thm If $\{f_n : E \subseteq \mathbb{C} \to \mathbb{C}\} \xrightarrow[\text{unif}]{} f$ and each f_n is continuous, then f is continuous.

Pf Take $z_0 \in E$. Given $\varepsilon > 0$ choose N s.t. $n \geq N \implies$ $|f(z) - f_n(z)| < \frac{\varepsilon}{3}$ $\forall z \in E$. Now choose $\delta > 0$ s.t. $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$ for $z \in E$, $|z - z_0| < \delta$. Then $z \in E$ and $|z - z_0| < \delta \implies$
$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

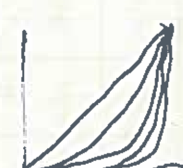
so $f(z) \longrightarrow f(z_0)$ as $z \to z_0$, i.e. f cts on E. $\square$

Thm If $\gamma : I \to \mathbb{C}$ is a path, $\{f_n : \gamma(I) \to \mathbb{C}\} \xrightarrow[\text{unif}]{} f$, f_n cts, then $\lim\limits_{n \to \infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz$.

Pf Given $\varepsilon > 0$, choose N s.t. $n \geq N$, $z \in \gamma(I) \implies |f(z) - f_n(z)| < \frac{\varepsilon}{\ell(\gamma)}$

Then, for $n \geq N$,
$$\left| \int_\gamma f - \int_\gamma f_n \right| = \left| \int_\gamma f - f_n \right| \leq \frac{\varepsilon}{\ell(\gamma)} \ell(\gamma) = \varepsilon. \quad \square$$

e.g. $\{|z|^n\} \xrightarrow[\text{pointwise}]{} \begin{cases} z \mapsto 1 & \text{if } |z| = 1 \\ z \mapsto 0 & \text{if } |z| < 1 \end{cases}$ on $\overline{D}_1(0)$



Convergence is not uniform on $\overline{D}_1(0)$.

Defn Say that an infinite series $\sum_{k=0}^{\infty} f_k(z)$ of fns defined on E converges uniformly on E if the sequence of partial sums converges uniformly on E.

Thm [Weierstrass M-test] For $\sum_{k=0}^{\infty} f_k(z)$ as above, if there is a convergent series of nonnegative #s $\sum_{k=0}^{\infty} M_k$ s.t. $|f_k(z)| \leq M_k$ for all k and all $z \in E$, then $\sum_{k=0}^{\infty} f_k$ converges uniformly on E.

Pf Comparison test gives pointwise convergence to some fn s. Let $s_n = \sum_{k=0}^{n} f_k$.
Then $|s(z) - s_n(z)| \leq \sum_{k=n+1}^{\infty} |f_k(z)| \leq \sum_{k=n+1}^{\infty} M_k$.
Since $\sum M_k$ converges, given $\varepsilon > 0$ may choose N s.t.
$n \geq N \Rightarrow \sum_{k \geq n+1} M_k < \varepsilon \Longrightarrow |s(z) - s_n(z)| < \varepsilon$ for $n \geq N$, $z \in E$. $\square$

e.g. $\left|\frac{z^k}{k^2}\right| \leq \frac{1}{k^2}$ for $|z| \leq 1$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ convergent implies $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges unif on $\overline{D}_1(0)$.

Radius of Convergence

Defn If $\{a_n\}$ is a sequence of real numbers, then $\limsup \{a_k\}$ is the limit of $\{u_n\}$ for $u_n = \sup\{a_k \mid k \geq n\}$.

Note $\{u_n\}$ is non-increasing, $\limsup\{a_n\}$ always well-defined in extended reals $[-\infty, \infty]$. $\{a_k\} \longrightarrow a \in [-\infty, \infty]$ iff $\limsup a_k = \liminf a_k = a$.

Thm Given a power series $\sum_{k=0}^{\infty} c_k (z-z_0)^k$, let

$$R = \frac{1}{\limsup |c_k|^{1/k}}.$$ Then the series converges absolutely

on $D_R(z_0)$ and diverges on $\mathbb{C} - \bar{D}_R(z_0)$. Furthermore, unif conv on $\bar{D}_r(z_0)$ $\forall r < R$.

Defn R is the radius of convergence of the series.

Pf WLOG, $z_0 = 0$. Let $u_n = \sup\{|c_k|^{1/k} \mid k \geq n\}$ so that $\limsup |c_k|^{1/k} = \lim u_n$. If $r < R$, choose $r < t < R$. Then

$t^{-1} > R^{-1} = \lim u_n \implies$ for $n \gg 0$, $u_n < t^{-1}$

$\implies$ for $k \geq n$, $|c_k|^{1/k} < t^{-1}$

$\implies$ for $k \geq n$, $|c_k| < t^{-k}$

If $|z| \leq r$, this implies $|c_k z^k| < \left(\frac{r}{t}\right)^k$ for $k \geq n$.

Since $\frac{r}{t} < 1$, $\sum_{k=n}^{\infty} (r/t)^k$ converges. By Weierstrass M, $\sum_{k=n}^{\infty} c_k z^k$ converges uniformly on $\bar{D}_r(0)$, and the same is true for $\sum_{k=0}^{\infty} c_k z^k$ since unif conv unaffected by finitely many terms. Unif conv on $\bar{D}_r(0)$ for $r < R$ $\implies$ absolut conv on $D_R(0)$.

Given $|z| > R$, $|z|^{-1} < \lim u_n \implies$ for each n there is $k > n$ with $|z|^{-1} < |c_k|^{1/k}$ so that $|c_k z^k| > 1$.

Thus $\{c_k z^k\} \not\to 0$ so the series diverges for $z \in \mathbb{C} - \bar{D}_R(0)$. $\square$

Cor Power series are continuous in their disc of convergence.

Prop If $\sum_{k \geq 0} c_k z^k$ has radius of conv R, then $\sum_{k \geq 1} c_k z^{k-1}$ has radius of conv R.

Pf Clearly $z \cdot \sum_{k\geq 1} kc_k z^{k-1} = \sum_{k\geq 0} kc_k z^k$ has the same radius of conv as $\sum_{k\geq 1} kc_k z^{k-1}$; call this R_1.

We have $R = \dfrac{1}{\limsup |c_k|^{1/k}}$

$$R_1 = \frac{1}{\limsup |kc_k|^{1/k}} = \frac{1}{\limsup k^{1/k} |c_k|^{1/k}}.$$

$\lim k^{1/k} = 1$, so $R_1 = R$. $\square$

Thm Let $f(z) = \sum_{k\geq 0} c_k (z-z_0)^k$ with radius of convergence R

Then $$\int_{z_0}^{z} f(w)\,dw = \sum_{k=0}^{\infty} \frac{c_k}{k+1} (z-z_0)^{k+1} \quad \forall z \in D_R(z_0).$$

Pf Let s_n be the nth partial sum of f.

Then
$$\begin{array}{ccc} \displaystyle\int_{z_0}^{z} s_n(w)\,dw & = & \displaystyle\sum_{k=0}^{n} \frac{c_k}{k+1}(z-z_0)^{k+1} \\ \Big\downarrow \text{unif} & & \Big\downarrow \text{unif} \\ \displaystyle\int_{z_0}^{z} f(w)\,dw & & \text{RHS} \end{array}$$

$\square$

Power Series Expansion

Thm Let $f(z) = \sum_{n\geq 0} c_n(z-z_0)^n$ with radius of conv R.

Then f is analytic on $D_R(z_0)$ and $f'(z) = \sum_{n\geq 1} nc_n(z-z_0)^{n-1}$

with radius of conv R.

Pf Let $g(z) = \sum_{n\geq 1} nc_n(z-z_0)^{n-1}$. Previously saw that g converges ~~uniformly on $D_R(z_0)$~~ has radius of conv R.

Know g is cts on $D_R(z_0)$ and $\int_{z_0}^{z} g(w) = \sum_{n=1}^{\infty} c_n(z-z_0)^n = f(z) - f(z_0)$.

This is an antideriv of g and $f(z_0)$ is constant, so $f' = g$. $\square$

e.g. Let $\mathrm{Log} = \log_{(-\pi,\pi]}$ be the principal branch of the logarithm. We know $\mathrm{Log}' = \frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$

Clearly $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$ has $\frac{1}{z}$ as its derivative as well. Thus $f' = \mathrm{Log}'$ and $f = C + \mathrm{Log}$.

But $f(0) = 1 = \mathrm{Log}(0)$, so $f = \mathrm{Log}$. $\square$

Cor If f has a power series exp'n about z_0 w/ radius of conv R, then it has ~~derivatives~~ of all orders on $D_R(z_0)$. Its k-th derivative is $f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!\, c_n}{(n-k)!} (z-z_0)^{n-k}$

and $f^{(k)}(z_0) = k!\, c_k$. $\square$

Cor If f has a power series exp'n about z_0 with positive radius of conv, then it has only one such expansion, and

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Power Series Expansions of Analytic Functions:

Thm Let f be analytic in an open set $U \subseteq \mathbb{C}$ and suppose $D_r(z_0) \subseteq U$ for some $r > 0$. Then there is a power series expansion for f, $f(z) = \sum_{n \geq 0} c_n (z - z_0)^n$, converging to $f(z)$ on $D_r(z_0)$.

Furthermore, $c_n = \frac{1}{2\pi i} \int_{|w - z_0| = s} \frac{f(w)}{(w - z_0)^{n+1}} \, dw$,

where s is any number with $0 < s < r$.

Pf If $0 < t < s < r$, $|w - z_0| = s$, $|z - z_0| \leq t$, then

$$\left| \frac{z - z_0}{w - z_0} \right| \leq \frac{t}{s} < 1$$

so
$$\frac{w - z_0}{w - z} = \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n \quad ⊛$$

with the final geometric series dominated by $\sum (t/s)^n$, which converges. By the M-test, ⊛ converges uniformly as a fn of $z \in D_t(z_0)$ and also of $w \in \partial D_s(z_0)$.

If we multiply ⊛ by $\frac{f(w)}{w - z_0}$ and integrate around $\partial D_s(z_0)$ to get

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_s(z_0)} \frac{f(w)}{w - z} \, dw$$

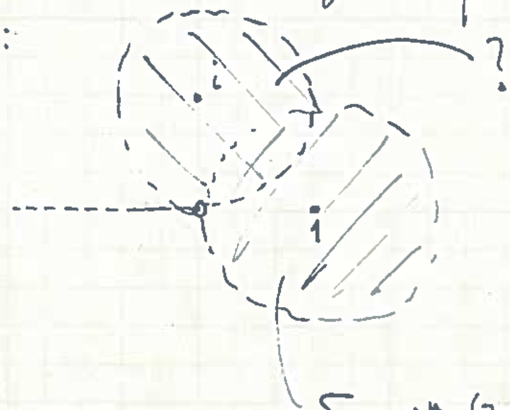
$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\int_{\partial D_s(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw \right) (z - z_0)^n$$

a power series exp'n for f on $D_t(z_0)$ with the appropriate coeffs. Since the coeffs don't depend on s, get conv on $D_r(z_0)$. $\square$

Cor If f is analytic on an open set U, then f has derivatives of all orders on U and they are all analytic. $\square$

<u>Note</u> Power series exp'n is local, on the largest open disc in U centered at a given point

<u>e.g.</u> Log:



$$\sum (-1)^n \frac{(z-1)^n}{n}$$

<u>Note</u> f analytic on open $U \supseteq \overline{D}_R(z_0)$. Then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\, dw .$$

<u>Cor</u> [Cauchy's Estimates] If f is analytic on open $U \supseteq \overline{D}_R(z_0)$, and if $|f(z_0)| \le M$ on $\partial D_R(z_0)$, then

$$|f^{(n)}(z_0)| \le \frac{n!\,M}{R^n}$$

for $n = 0, 1, 2, \ldots$

<u>Pf</u>
$$\left| \frac{f(w)}{(w-z_0)^{n+1}} \right| \le \frac{M}{R^{n+1}} \quad \text{on } \partial D_R(z_0).$$

$$\Longrightarrow |f^{(n)}(z_0)| \le \frac{n!}{2\pi} \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n!\,M}{R}. \quad \square$$

<u>Thm</u> $\{f_n : U \subseteq \mathbb{C} \xrightarrow[\text{analytic}]{} \mathbb{C}\} \xrightarrow[\substack{\text{on compact}\\ K \subseteq U}]{\text{unif}} f$. Then f is analytic on U.

<u>Pf</u> Read pp. 87–88. $\square$

Liouville's Thm

Defn A function analytic on all of $\mathbb{C}$ is called entire.

Thm [Liouville] The only bounded entire functions are the constant functions.

Pf By the Cauchy estimates, if $|f(z)| \le M$ $\forall z \in D_R(z_0)$, then $|f'(z_0)| \le \frac{M}{R}$. If f is bounded by M on $\mathbb{C}$, then this holds for all $R > 0$. Taking $R \to \infty$, get $f'(z_0) = 0$. Hence f is constant. □

Thm If f is defined and cts on $\mathbb{C}$ and $\lim_{z\to\infty} f(z)$ exists, then f is bounded on $\mathbb{C}$.

Here $\lim_{z\to\infty} f(z) = L \in \mathbb{C}$ means $\forall \varepsilon > 0$ $\exists R > 0$ st $|z| > R$ $\Rightarrow |f(z) - L| < \varepsilon$.

Pf If $\lim_{z\to\infty} f(z) = L$, then $\exists R > 0$ s.t. $|f(z) - L| < 1$ for $|z| > R$.

Thus $|f(z)| < |L| + 1$ if $|z| > R$. f cts on $\mathbb{C}$, $\overline{D}_R(0)$ compact $\Rightarrow f$ bdd on $\overline{D}_R(0)$ hence bdd on $\mathbb{C}$. □

Fundamental Theorem of Algebra Every nonconstant complex polynomial has a complex root.

Pf Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a poly of deg $n \ge 1$ so $a_n \ne 0$. Assume for contradiction that p has no root in $\mathbb{C}$, so $p(z) \ne 0$ $\forall z \in \mathbb{C}$. Then $\frac{1}{p}$ is an entire function, and we will show it is also bounded. Let $h(z) = \frac{z^n}{p(z)} = \frac{1}{a_n + a_{n-1} z^{-1} + \dots + a_1 z^{1-n} + a_0 z^{-n}}$.

Then $\frac{1}{p(z)} = \frac{h(z)}{z^n}$ for $z \neq 0$. Furthermore, $\lim_{z\to\infty} h(z) = 1/a_n$, so $\lim_{z\to\infty} \frac{1}{p(z)} = \lim_{z\to\infty} \frac{h(z)}{z^n} = 0$. Thus $1/p$ is bounded on all of $\mathbb{C}$, so Liouville's Thm implies $1/p$ is constant, ⚡. $\square$

<u>Cor</u> Each complex polynomial factors completely into constant and monic linear factors.

<u>Cor</u> Every $A \in M_{n\times n}(\mathbb{C})$ has at least one cpx eigenvalue.

<u>Cor</u> We can solve a bunch of differential equations.

etc etc!

<u>Thm</u> An entire function f is a polynomial of degree $\leq n$ iff $\exists A, B > 0$ s.t. $|f(z)| \leq A + B|z|^n \ \forall z \in \mathbb{C}$.

<u>Pf</u> Suppose $p(z) = a_n z^n + \cdots + a_0$. Then $\lim_{z\to\infty} \frac{p(z)}{z^n} = a_n$

so $\exists R > 0$ s.t. $|z| > R \Rightarrow \left| \frac{p(z)}{z^n} - a_n \right| < 1$

$$\Rightarrow \frac{|p(z)|}{|z^n|} < |a_n| + 1$$

$$\Rightarrow |p(z)| < \underbrace{(|a_n|+1)}_{B} |z|^n$$

Take $A > 0$ s.t. $|p(z)| \leq A$ on $\overline{D}_R(0)$ (by EVT).

Then $|p(z)| \leq A + B|z|^n$ on $\mathbb{C}$.

Converse: <u>HW 6</u> via Cauchy's estimates. $\square$