Properties of the Index Function

Goal: $\mathrm{Ind}_\gamma : \mathbb{C} \setminus \gamma(I) \to \mathbb{Z}$ is constant on the connected components of $\mathbb{C} \setminus \gamma(I)$

Connected Sets

Defn A set $E \subseteq \mathbb{C}$ is separated if $\exists$ a pair A, B of open subsets of $\mathbb{C}$ s.t. $E \subseteq A \cup B$, $A \cap E, B \cap E \neq \emptyset$, $A \cap B = \emptyset$. Say that A, B separate E. If E is not separated, then call it connected.

A maximal connected subset containing $z \in E$ is called the (connected) component of z. Two conn'd components of E are identical or disjoint (why?) so conn'd components partition E.

Call $E \subseteq \mathbb{C}$ path connected if every two points in E can be joined with a path in E.

Thm Let $U \subseteq \mathbb{C}$ be open. Then

(a) each component of U is open

(b) U is connected iff U is path connected.

Pf (a) Let $V \subseteq U$ be a conn'd component containing z. Then $V = \bigcup$ conn'd subsets containing z. Since U open, $\exists r > 0$ with $D_r(z) \subseteq U$. Since $D_r(z)$ conn'd, $D_r(z) \subseteq V$, so V open.

(b) Suppose U conn'd. For $z \in U$, let V_z = pts of U conn'd to z by a path in U. Let $w \in U$. $\exists$ open disc $D = D_r(w) \subseteq U$. Either $D \subseteq V_z$ or $D \subseteq U \setminus V_z$. $\Rightarrow$ V_z, $U \setminus V_z$ open $\subseteq U$ with union U. Since U conn'd, one of them is empty. $z \in V_z \neq \emptyset$ $\Rightarrow U \setminus V_z = \emptyset \Rightarrow V_z = U$ so U path conn'd

Suppose U path conn'd, sep'd by A, B. Then
$f: U \to \mathbb{C}$, $u \mapsto \begin{cases} 1 & u \in A \\ 0 & u \in B \end{cases}$ cts. Since U is path conn'd, $\exists$ path γ
connecting $a \in A$ to $b \in B$. Then $f \circ \gamma$ cts & IVT. $\square$

<u>Note</u> • For arbitrary spaces, path conn'd $\Rightarrow$ conn'd, but not the converse (topologist's sine curve

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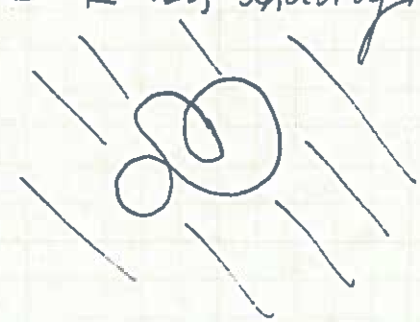
• For K compact, $\mathbb{C} \setminus K$ is the union of its conn'd cpts, each of which is open and path conn'd.

<u>Thm</u> If $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \setminus K$ has exactly one unbounded component.



<u>Pf</u> $K \subseteq \overline{D}_R(0)$ since closed and bdd.

$\Rightarrow \underbrace{\mathbb{C} \setminus K}_{\text{open}} \supseteq \underbrace{\mathbb{C} \setminus \overline{D}_R(0)}_{\text{connected hence contained in a component of } \mathbb{C} \setminus K}$.

$\Rightarrow$ all other components of $\mathbb{C} \setminus K$ contained in $\overline{D}_R(0)$ hence bounded. $\square$

<u>Thm</u> If $\gamma: I \to \mathbb{C}$ is a closed path, then $\mathrm{Ind}_\gamma(z)$ is constant on each component of $\mathbb{C} \setminus \gamma(I)$, and is 0 on the unbounded component.

<u>Pf</u> Take $z_0 \in \mathbb{C} \setminus \gamma(I)$ and $D_R(z_0) \subseteq \mathbb{C} \setminus \gamma(I)$. First show that on some smaller disc centered at z_0, Ind_γ is constant.
Suppose $0 < r < R$, $z \in D_r(z_0)$. Then

$$\mathrm{Ind}_\gamma(z) - \mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} - \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z_0}$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{z - z_0}{(w - z)(w - z_0)} \, dw.$$

For $w\in\gamma(I)$, $|w-z_0|\ge R$, $|w-z|\geq R-r$.

Since $|z-z_0|<r$, $\left|\dfrac{z-z_0}{(w-z)(w-z_0)}\right| \le \dfrac{r}{R(R-r)}$

$\Rightarrow \quad |Ind_\gamma(z)-Ind_\gamma(z_0)| \le \dfrac{r\,\ell(\gamma)}{2\pi R(R-r)} < 1$ for r suff small

But & $\in\mathbb{Z}$, so in fact are equal on $D_r(z_0)$.

Let A be a component of $\mathbb{C}-\gamma(I)$, and for each $n\in\mathbb{Z}$ let $V_n = \{z\in A \mid Ind_\gamma(z)=n\}$. Each $V_n \subseteq A$ open, by above. $\bigcup_{m\neq n} V_m$ open as well with $V_n \cup \bigcup_{m\neq n} V_m = A$. Since A is conn'd, one of the sets is empty. Thus $V_n\neq\emptyset \Rightarrow V_n = A$ and Ind_γ constant on components of $\mathbb{C}-\gamma(I)$.

Remains to show $Ind_\gamma(z)=0$ on unbdd cpt of $\mathbb{C}-\gamma(I)$.

Take D open disc $\supseteq\gamma(I)$, $z_0\in\mathbb{C}-D$, so z_0 in unbdd cpt.

$$Ind_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z_0} = 0 \text{ since } D \text{ convex } \supseteq\gamma(I)$$

with $\dfrac{1}{w-z_0}$ analytic on D. □

e.g. γ tracing n times around z_0 in circle of radius r:

$$\gamma : [0,2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto z_0 + re^{int}.$$

Then $$Ind_\gamma(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)-z_0}\,dt$$
$$= \frac{1}{2\pi i}\int_0^{2\pi} in\,dt = n.$$

$Ind_\gamma(V_{z_0}) = \{n\}$, $Ind_\gamma(\mathbb{C}-\overline{V}_{z_0}) = \{0\}$.

Defn $\gamma : [a,b] \to \mathbb{C}$ path, D open disc. Say γ simply splits D if

(a) $J = \gamma^{-1}(D) = (c,d) \subseteq [a,b]$ or, in case γ closed with $\gamma(a) = \gamma(b) \in D$, $J = [a,c) \cup (d,b] \subseteq [a,b]$ with $c < d$.

(b) $D \setminus \gamma(J)$ has exactly two components

Thm Let γ be a closed path which simply splits a disc D. Then $\mathrm{Ind}_\gamma(z) = 1 + \mathrm{Ind}_\gamma(w)$ if z is in the left and w in the right component of $D \setminus \gamma(J)$.

e.g.