Properties of Contour Integrals

Thm [Independence of Parametrization]

Let $\gamma_1 : [a,b] \to \mathbb{C}$ be a path and $\alpha : [c,d] \to [a,b]$ a smooth function with $\alpha(c) = a$, $\alpha(d) = b$. If $\gamma_2 = \gamma_1 \circ \alpha$, then

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

for all f defined and cts on a set containing $\gamma_1([a,b]) = \gamma_2([c,d])$.

Pf By chain rule, $\gamma_2' = (\gamma_1' \circ \alpha) \cdot \alpha'$. Thus

$$\int_{\gamma_2} f = \int_c^d (f \circ \gamma_2) \cdot \gamma_2'$$

$$= \int_c^d (f \circ \gamma_1 \circ \alpha) \cdot (\gamma_1' \circ \alpha) \cdot \alpha'$$

$$= \int_a^b (f \circ \gamma_1) \cdot \gamma_1' \qquad [\text{change of variables, } u = \alpha]$$

$$= \int_{\gamma_1} f . \qquad \square$$

(2) Ind of param'n, but not of image.

e.g. $\gamma_1(t) = re^{it}$, $\gamma_2(t) = re^{-it}$ both on $[0, 2\pi]$.

Then $\displaystyle\int_{\gamma_1} \frac{dz}{z} = 2\pi i$, $\displaystyle\int_{\gamma_2} \frac{dz}{z} = -2\pi i$.

Need $\alpha(c) = a$, $\alpha(d) = b$!

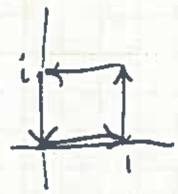
TPS What are the reasonable interpretations of

$$\int_{|z|=1} f, \quad \int_{\partial\Delta} f, \quad \int_{w_1}^{w_2} f$$

for $\Delta \subseteq \mathbb{C}$ triangle, $w_1, w_2 \in \mathbb{C}$? (Preferred or'n is counterclockwise

Defn A closed curve (or path) γ is one such that $\gamma(a)=\gamma(b)$ (for $\gamma:[a,b]\longrightarrow\mathbb{C}$).

Preview of Cauchy's Thm If $\gamma: I\to U\subseteq\mathbb{C}$ is a closed path, f is analytic on U, and γ does not "go around any holes in U," then $\int_\gamma f=0$.

e.g. $f(z)=z$, γ:  $\int_\gamma f=0$.

Operations on Contour Integrals

$\gamma_1+\gamma_2$

Join of two paths when γ_1 ends where γ_2 starts

Simple enough, but what if the parameter intervals don't match up?

A $\alpha:[c,d]\longrightarrow[a,b]$ is smooth with $\alpha(c)=a$, $\alpha(d)=b$

$t\longmapsto a+\dfrac{b-a}{d-c}(t-c)$

Thus we can always reparametrize to get whatever parameter interval we want.

Now if $\gamma_1:[a,b]\to\mathbb{C}$, $\gamma_2:[b,c]\to\mathbb{C}$ with $\gamma_1(b)=\gamma_2(b)$,

define $$\gamma_1+\gamma_2\,(t)=\begin{cases}\gamma_1(t) & \text{if } t\in[a,b]\\ \gamma_2(t) & \text{if } t\in[b,c].\end{cases}$$

Define the <u>reverse</u> of $\gamma : [a,b] \to \mathbb{C}$ by

$$-\gamma : [a,b] \longrightarrow \mathbb{C} \qquad\qquad -\gamma(t) \neq -(\gamma(t))$$
$$t \longmapsto \gamma(a+b-t)$$

For closed simple paths, call counterclockwise orientation to be <u>positive</u>; clockwise to be <u>negative</u>.

<u>Thm</u> $\gamma, \gamma_1, \gamma_2$ paths, γ_1 ending where γ_2 starts, f, g cts fns $\Omega \to \mathbb{C}$ with $\operatorname{im}\gamma, \operatorname{im}\gamma_1, \operatorname{im}\gamma_2 \subseteq \Omega$, $\lambda \in \mathbb{C}$. Then

(a) $\int_\gamma \lambda f + g = \lambda \int_\gamma f + \int_\gamma g$

(b) $\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

(c) $\int_{-\gamma} f = -\int_\gamma f$

<u>Length</u>

<u>Defn</u> The <u>length</u> of $\gamma : [a,b] \to \mathbb{C}$ is $\ell(\gamma) = \int_a^b |\gamma'(t)|\, dt$

<u>Thm</u> $\gamma : I \to \mathbb{C}$ path, $f : \Omega \to \mathbb{C}$ cts with $\gamma(I) \subseteq \Omega$, $|f(z)| \leq M$ $\forall z \in \gamma(I)$, then

$$\left| \int_\gamma f \right| \leq M \ell(\gamma).$$

<u>Pf</u> $$\left| \int_\gamma f \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\, dt \right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, dt$$
$$\leq \int_a^b M |\gamma'(t)|\, dt = M \ell(\gamma). \qquad \square$$

Cauchy's Integral Thm for a Triangle

Idea  $U \xrightarrow[\text{analytic}]{f} \mathbb{C} \Rightarrow \int_\gamma f = 0$

for U "simply connected."

First case: $U = \Delta$

Note: $\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$ if $F' = f$ (HW)

so $\int_\gamma f = 0$ if γ closed and f has a primitive F.

Idea: cpx diff'l fns have linear approx'ns, and linear functions have primitives, so analytic fns are approximated by fns w/ integral 0 around closed paths.

Lemma Let f be cts on a nbhd of $w \in \mathbb{C}$, f cpx diff'l at w. Then $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.
$$\left| \int_{\partial\Delta} f \right| < \varepsilon d^2$$
if Δ is any triangle containing w of diameter $d \le \delta$.

Note: $\operatorname{diam}(\Delta)$ = longest side length.

Pf Since f is cts on a nbhd of w, $\exists r > 0$ s.t. f is cts on $D_r(w)$.

Know that $\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = f'(w)$ exists, so $\exists\ 0 < \delta < r$ s.t.
$$|z - w| < \delta \implies \left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \frac{\varepsilon}{3}$$
$$\implies |f(z) - f(w) - f'(w)(z - w)| < \frac{\varepsilon}{3}|z - w| \quad \text{for } z \in D_\delta(w).$$

For Δ a triangle of diam $d \le \delta$, containing w, set
$$I = \int_{\partial\Delta} f$$

Then

$$I = \underbrace{\int_{\partial\Delta} \big(f(w) + f'(w)(z-w)\big)\,dz}_{\substack{=0 \text{ b/c linear} \\ \text{in } z \text{ so has a} \\ \text{primitive}}} + \int_{\partial\Delta} \big(f(z) - f(w) - f'(w)(z-w)\big)\,dz$$

so $$I = \int_{\partial\Delta} \big(f(z) - f(w) - f'(w)(z-w)\big)\,dz$$

$$\Rightarrow |I| \le \int_{\partial\Delta} |f(z) - f(w) - f'(w)|\,dz$$

$$< \int_{\partial\Delta} \frac{\varepsilon}{3}|z-w|\,dz$$

$$< \int_{\partial\Delta} \frac{\varepsilon d}{3}\,dz$$

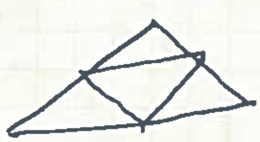
$$\le \frac{\varepsilon d}{3}(3d) = \varepsilon d^2. \qquad \square$$

<u>Thm</u> Let f be a function which is analytic in an open set U, and suppose Δ is a triangle contained in U. Then

$$\int_{\partial\Delta} f = 0.$$

<u>Pf</u> Set $I = \int_{\partial\Delta} f$. We show $I = 0$ by showing $|I| < \varepsilon$ $\forall \varepsilon > 0$.

Let $\varepsilon > 0$. Subdivide Δ into four triangles via side midpts:



Get all subtriangles similar to Δ.

Have

Let Δ_1 be a subtriangle s.t. $|I_1| \geq |I|/4$ for $I_1 = \int_{\partial\Delta_1} f$

Note that if $\operatorname{diam}(\Delta) = h$ then $\operatorname{diam}(\Delta_1) = \frac{h}{2}$.

Repeat the subdivision with Δ_1 to get Δ_2 of diameter $\frac{h}{2^2}$ and with $|I_2| \geq |I|/4^2$, $I_2 = \int_{\partial\Delta_2} f$.

Proceeding by induction, get Δ_n of diameter $\frac{h}{2^n}$ with $|I_n| \geq |I|/4^n$ (★), $I_n = \int_{\partial\Delta_n} f$.

Then $\Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots$ is a nested sequence of compact subsets of $\mathbb{C}$ $\Rightarrow \exists\, w \in \bigcap_{n\geq 1} \Delta_n$.

By the lemma (with $\frac{\varepsilon}{h^2}$ in place of ε), we conclude that $\exists\, \delta > 0$ s.t. the integral of f around any triangle containing w of diameter $d \leq \delta$, is less than $d^2\varepsilon/h^2$.

Now take $n \gg 0$ so that $h_n = \frac{h}{2^n} < \delta$. Then

$$|I_n| < \frac{h_n^2}{h^2}\varepsilon = \frac{\varepsilon}{4^n}$$

Combined with (★), $|I| \leq 4^n |I_n| < \varepsilon$. Hence $I = 0$. $\square$

<u>Thm</u> The same, but f cts on U, analytic on $U \setminus \{c\}$ for some exceptional pt $c \in \Delta$.

<u>Pf</u> Let $\varepsilon > 0$. If c is a vx, subdivide Δ into smaller & smaller triangles in such a way that the one containing c has circumference $< \frac{\varepsilon}{M}$, M = max'l value of $|f|$ on Δ. $|\int_{\Delta'} f| < \varepsilon$ for $c \in \Delta'$.

$\int_{\text{other } \Delta\text{'s}} f = 0$ by previous thm so $|\int_\Delta f| < \varepsilon$ $\forall \varepsilon > 0$. ✓

Other cases:

$\square$

Cauchy's Thm for a convex set

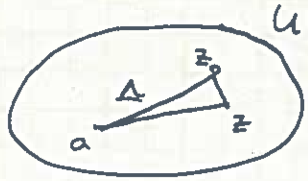
Defn $C \subseteq \mathbb{C}$ is convex if $\forall a, b \in C$, the line segment joining a & b is contained in C:

Thm Let U be a convex open set and suppose f is a cts function on U and has the property that $\int_{\partial\Delta} f = 0$ for all triangles $\Delta \subseteq U$. If $a \in U$ is fixed and

$$F : U \longrightarrow \mathbb{C}, \quad z \longmapsto \int_a^z f(w)\,dw$$

then $F' = f$ on U.

Pf For $z, z_0 \in U$ consider



By convexity, $\Delta \subseteq U$.

Take $\partial\Delta$ to be the path a to z to z_0 to a. Then

$$0 = \int_{\partial\Delta} f = \int_a^z f + \int_z^{z_0} f + \int_{z_0}^a f = F(z) - F(z_0) - \int_{z_0}^z f.$$

Thus
$$F(z) - F(z_0) = \int_{z_0}^z f = \int_{z_0}^z f(z_0)\,dw + \int_{z_0}^z (f(w) - f(z_0))\,dw$$
$$= f(z_0)(z - z_0) + \int_{z_0}^z (f(w) - f(z_0))\,dw$$

$$\Longrightarrow \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{z_0}^z (f(w) - f(z_0))\,dw$$

To show $F'(z_0) = f(z_0)$, need to show RHS $\longrightarrow 0$ as $z \to z_0$.
Let $\varepsilon > 0$. By continuity, $\exists \delta > 0$ s.t. $|f(w) - f(z_0)| < \varepsilon$ when $|w - z_0| < \delta$.
If $|z - z_0| < \delta$, then $|w - z_0| \leq \delta$ $\forall w \in [z, z_0]$ so $|f(w) - f(z_0)| < \varepsilon$ $\forall w \in [z, z_0]$.

Then $\left| \int_{z_0}^{z} (f(w)-f(z_0))\,dw \right| < \varepsilon |z-z_0|$

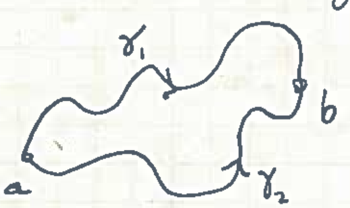
$$\Rightarrow \left| \frac{1}{z-z_0}\int_{z_0}^{z}(f(w)-f(z_0))\,dw \right| < \varepsilon \quad \text{for } |z-z_0|<\delta$$

Thus $\lim_{z\to z_0} \frac{1}{z-z_0}\int_{z_0}^{z}(f(w)-f(z_0))\,dw = 0$, as desired. $\square$

<u>Thm</u> Let U be a convex open set and suppose f is analytic on U, except possibly at one pt, and cts on U. Then $\int_\gamma f = 0$ for all closed paths γ in U.

<u>Pf</u> f has an ~~antiderivative~~ primitive fn on U. $\square$

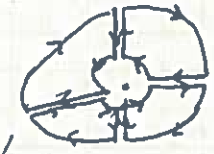
<u>Cor</u> (TPS) If U is convex open, f analytic on U, $a,b\in U$, then $\int_\gamma f$ is the same for all γ starting at a, ending at b.



$\int_{\gamma_1-\gamma_2} f = 0 \ldots$

<u>Prop</u> / <u>e.g.</u> $\int_\gamma \frac{dz}{z} = 2\pi i$ for

<u>Pf</u>



$$\Rightarrow \int_\gamma \frac{dz}{z} = \int_{\odot} \frac{dz}{z} = 2\pi i.$$

(each piece inside a convex open set on which $\frac{1}{z}$ is analytic $\Rightarrow \int \frac{dz}{z} = 0$ around these. $\square$

<u>Index</u> (of a path around a point)

<u>Defn</u> $\gamma: I \to \mathbb{C}$ any closed path in $\mathbb{C}$, $z\in\mathbb{C}\setminus\gamma(I)$. The <u>index</u> of z wrt γ is $\mathrm{Ind}_\gamma(z) := \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}$.

Thm If γ is a closed path in $\mathbb{C}$ with parameter interval $I=[a,b]$ then $\mathrm{Ind}_\gamma(z)$ is an integer-valued fn of $z \in \mathbb{C} \setminus \gamma(I)$.

Pf Take $z_0 \in \mathbb{C}\setminus\gamma(I)$. Have $\gamma(a)=\gamma(b)$. Define $\lambda : I \to \mathbb{C}$ by
$\lambda(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s)-z_0}\,ds$. Then $\lambda(a)=0$ and $\lambda(b) = 2\pi i\,\mathrm{Ind}_\gamma(z)$.

Suffices to show $e^{\lambda(b)}=1$ (b/c this gives $\lambda(b) = 2\pi i n$, $n \in \mathbb{Z}$).

By FTC, $\lambda'(t) = \frac{\gamma'(t)}{\gamma(t)-z_0}$ while

$$\left(e^{\lambda(t)}\right)' = e^{\lambda(t)}\lambda'(t) = e^{\lambda(t)}\frac{\gamma'(t)}{\gamma(t)-z_0}$$

Thus $$\left(\frac{e^{\lambda(t)}}{\gamma(t)-z_0}\right)' = \frac{1}{(\gamma(t)-z_0)^2}\left(e^{\lambda(t)}\frac{\gamma'(t)}{\gamma(t)-z_0}(\gamma(t)-z_0) - e^{\lambda(t)}\gamma'(t)\right) = 0$$

Hence $\frac{e^{\lambda(t)}}{\gamma(t)-z_0}$ is constant. In particular, $\frac{e^{\lambda(t)}}{\gamma(t)-z_0} = \frac{e^{\lambda(a)}}{\gamma(a)-z_0}$
$= \frac{1}{\gamma(a)-z_0}$ $\forall t \in [a,b]$. Setting $t=b$, get

$$e^{\lambda(b)} = \frac{\gamma(b)-z_0}{\gamma(a)-z_0} = 1, \text{ as desired.} \qquad \square$$

Thm $U \subseteq \mathbb{C}$ convex open, $f : U \to \mathbb{C}$ analytic, $\gamma : I \to U$ closed path. Then $\mathrm{Ind}_\gamma(z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$ $\forall z \in U \setminus \gamma(I)$.

Pf Define $g : U\times U \to \mathbb{C}$ by $g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{o/w} \end{cases}$

For any $z \in U$, $0 = \int_\gamma g(z,w)\,dw = \int_\gamma \frac{f(w)}{w-z}\,dw - \int_\gamma \frac{f(z)}{w-z}\,dw$

$$= \int_\gamma \frac{f(w)}{w-z}\,dw - 2\pi i\,\mathrm{Ind}_\gamma(z)\,f(z)$$

TPS Why does $g(z,-)$ satisfy Cauchy's Thm hypotheses? $\square$

Interpretation For $\mathrm{Ind}_\gamma(z) \neq 0$ (z "inside" γ), values of f at z determined by values of f ~~on~~ path.