**Continuity** $f: \Omega \to \mathbb{C}$ is *continuous* if either of the following equivalent conditions holds:

① For all $a \in \Omega$, $\lim_{z \to a} f(z) = f(a)$

② For all open $W \subseteq \mathbb{C}$, $f^{-1}(W) \subseteq \Omega$ is open.

Reading: 2.1 (review since equivalent to continuity of function $\Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$).

## The Complex Derivative

**Defn** Let $f$ be a fn defined on a nbhd of $z \in \mathbb{C}$. If $\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ exists, we denote it $f'(z)$ and say $f$ is *complex diff'l* at $z$ with *complex derivative* $f'(z)$. If $f$ is defined and diff'l at every point of an open set $U$, then call $f$ *analytic* on $U$.

e.g. · For $f(z) = z$, $\lim_{h \to 0} \frac{z + h - z}{h} = \lim_{h \to 0} 1 = 1$, so $f'(0) = 1$.

· For $g(z) = \bar{z}$, $\lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h \to 0} e^{-2i\theta}$ for $h = re^{i\theta}$. This limit does not exist! (Different values along ~~any~~ ray emanating from 0.) So $g$ is not complex differentiable at $z = 0$, despite $G: \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, -y)$ being infinitely diff'l!

**Prop** exp is analytic on $\mathbb{C}$ and $\exp' = \exp$.

Pf Have $\dfrac{e^{z+h}-e^z}{h} = e^z\,\dfrac{e^h-1}{h}$, so suffices to show

$\dfrac{e^h-1}{h} \to 1$ as $h \to 0$. Expanding its power series gives

$$\frac{e^h-1}{h}-1 = \frac{e^h-1-h}{h} = \frac{\frac{h^2}{2}+\frac{h^3}{3!}+\cdots}{h} = \frac{h}{2}+\frac{h^2}{3!}+\cdots \to 0,$$

so $(e^z)' = e^z$. $\square$

__Basic properties__

- If $f'(a)$ exists, then $f$ is cts at $a$.
- $(f+g)'(z) = f'(z)+g'(z)$ when RHS makes sense.
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ — " —
- If $g(z) \neq 0$ and $g'(z)$ exists, then $(1/g)'(z) = -g'(z)/g^2(z)$.
- $(f/g)'(z) = \dfrac{f'(z)g(z)-g'(z)f(z)}{g^2(z)}$
- $(f\circ g)'(a) = f'(g(a))\,g'(a)$

Proofs exactly mirror those from Math 112.

__Cauchy-Riemann Equations__

For $f:\Omega \to \mathbb{C}$ write $f(x+iy) = u(x,y)+iv(x,y)$ to consider $f$ a function on a subset of $\mathbb{R}^2$ with components $u,v$.

Complex diff'lity ~~Analyticity~~ at $z_0$ is equivalent to the existence of $c = a+ib$ s.t.

① $$\lim_{z\to z_0} \frac{1}{z-z_0}\left(f(z)-f(z_0)-c(z-z_0)\right) = 0.$$

The function $F: (x,y) \mapsto (u(x,y), v(x,y))$ is diff'l at $(x_0,y_0)$ iff $\exists\, M = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ s.t.

(2) $$\lim_{(x,y)\to(x_0,y_0)} \frac{1}{|(x-x_0, y-y_0)|} \left| F(x,y) - F(x_0,y_0) - M\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \right| = 0.$$

It might (but may not) happen that

(3) $$M = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \in \mathbb{C} \quad (\text{so } r = -q,\ s = p)$$

From HW, you know that if $M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then (for $c = a+ib$)

$$cz = (ax-by) + i(bx+ay) \longleftrightarrow \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$$

Thus (1) holds iff (2) & (3) hold (and in this case, $p = a$, $q = b$).

<u>Thm</u> [Cauchy-Riemann equations]

$f: \Omega \to \mathbb{C}$ is ~~analytic~~ cpx diff'l at $z_0$ iff $F$ "$(u,v)$" is diff'l at $(x_0,y_0)$ with $\boxed{u_x = v_y,\ u_y = -v_x}$ at $(x_0,y_0)$.

In this case, $f' = u_x + iv_x = v_y - iu_y$. $\square$

<u>e.g.</u> With $z = x+iy$, $e^z = e^x(\cos y + i\sin y)$ so $u = e^x\cos y$, $v = e^x \sin y$. Further,

$$u_x = e^x \cos y = v_y$$
$$u_y = -e^x \sin y = -v_x$$

as expected since $e^z$ is analytic.

## Harmonic Functions

In a bit, we'll prove that analytic fns have cts cpx derivatives of all orders. Assuming this result for a moment, we have

**Thm** If $f:\Omega \to \mathbb{C}$ has $f = u + iv$ and is analytic on $\Omega$, then $\underbrace{u_{xx} + u_{yy} = 0}_{\text{Say } u \text{ is harmonic}}$. and $v_{xx} + v_{yy} = 0$.

**Pf** $u_{xx} = (u_x)_x \underset{CR}{=} (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}$

$v_{xx} = (v_x)_x \underset{CR}{=} (-u_y)_x = -(u_x)_y = -(v_y)_y = -v_{yy}$.

**Defn** If $u, v$ are harmonic functions s.t. $f = u + iv$ is analytic, then we call $u, v$ harmonic conjugates of one another.

**Facts**
- If it exists, the harmonic conjugate of $u$ is unique up to an additive constant
- $u$ is conjugate to $v$ iff $v$ is conjugate to $u$.

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$\boxed{2F}$

## Contour integrals

A curve (or contour) in $\mathbb{C}$ is a continuous function
$$\gamma : I \to \mathbb{C} \quad \text{where} \quad I = [a,b] \subseteq \mathbb{R} \text{ is an interval.}$$
For $c \in I$, define $\gamma'(c) = \lim_{t \to c} \frac{\gamma(t) - \gamma(c)}{t - c}$. If $\gamma(t) = x(t) + iy(t)$, then $\gamma'(c) = x'(c) + iy'(c)$.

Call $\gamma$ continuously differentiable on $I$ if diff'l at all $c \in I^\circ$ with $\gamma'$ cts. In this case, write $\gamma \in C^1(I)$.

$\gamma$ is <u>piecewise smooth</u> if $\exists\ a = a_0 < a_1 < \cdots < a_n = b$ s.t. $\gamma$ is $C^1$ on each $[a_{j-1}, a_j]$. A curve that is piecewise smooth will be called a <u>path</u>.

<u>e.g.</u> $\cdot$ $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$
$t \longmapsto re^{it}$

$\cdot$ $\gamma : [0,1] \longrightarrow \mathbb{C}$
$t \longmapsto (1-t)z + tw$ for some fixed $z, w \in \mathbb{C}$

<u>TPS</u> What do these parametrize?

Riemann Integral of cpx-valued Fns:

$f : I \longrightarrow \mathbb{C}$
$t \longmapsto g(t) + ih(t)$ for $g, h$ real valued, $I = [a,b]$

define $\int_a^b f = \int_a^b g + i\int_a^b h$.

<u>Thm</u> The Riemann integral of cpx fns is $\mathbb{C}$-linear. $\square$

<u>Thm</u> $\int_a^b f = \int_a^c f + \int_c^b f$. $\square$

<u>Thm</u> $\left|\int_a^b f\right| \le \int_a^b |f|$

<u>Pf</u> (by trick!) Set $w = \int_a^b f \in \mathbb{C}$. If $w = 0$, done. If $w \neq 0$, set $u = \frac{\overline{w}}{|w|}$, so $uw = |w|$. Thus

$$|w| = \left|\int_a^b f\right| = u\int_a^b f = \int_a^b uf \in \mathbb{R}.$$ Thus $\int_a^b \operatorname{Im}(uf) = 0$

and $\left|\int_a^b f\right| = \int_a^b \operatorname{Re}(uf) \le \int_a^b |uf| = \int_a^b |f|$ since $|u| = 1$. $\square$

Integration along a path:

If $f : \Omega \to \mathbb{C}$ continuous and $\gamma(I) \subseteq \Omega$, then $(f \circ \gamma) \cdot \gamma'$ is defined on $I$ (except at finitely many discontinuities of $\gamma'$), and is piecewise cts, hence Riemann integrable.

<u>Defn</u> For $\gamma : [a,b] \to \mathbb{C}$ a path, $f : \Omega \to \mathbb{C}$ cts with $\gamma([a,b]) \subseteq \Omega$, the <u>integral</u> of $f$ over $\gamma$ is

$$\int_\gamma f := \int_a^b (f \circ \gamma) \cdot \gamma' .$$

$$\left( \text{or } \int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt \right)$$

Think: $z \rightsquigarrow \gamma(t)$

$dz \rightsquigarrow \gamma'(t)\, dt$ .

<u>e.g.</u> $f(z) = z$

$\gamma : [0, 2\pi] \to \mathbb{C}$ $\qquad \gamma'(t) = i r e^{it}$

$t \longmapsto re^{it}$

$$\int_\gamma z\, dz = \int_0^{2\pi} re^{it} \cdot ire^{it}\, dt$$

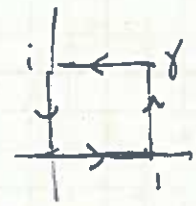
$$= ir^2 \int_0^{2\pi} e^{2it}\, dt$$

$$= ir^2 \int_0^{2\pi} \left( \cos(2t) + i\sin(2t) \right) dt$$

$$= 0 .$$

e.g. $\gamma : [0,4] \longrightarrow \mathbb{C}$

$$t \longmapsto \begin{cases} t & \text{if } t \in [0,1) \\ 1+i(t-1) & \text{if } t \in [1,2] \\ 1-(t-2)+i & \text{if } t \in [2,3) \\ (1-(t-3))i & \text{if } t \in [3,4) \end{cases}$$




$\int_\gamma x\, dz$ :

$$\int_0^1 Re(\gamma)\gamma' = \int_0^1 t\, dt = \frac{1}{2}$$

$$\int_1^2 Re(\gamma)\gamma' = \int_1^2 i\, dt = i$$

$$\int_2^3 Re(\gamma)\gamma' = \int_2^3 (3-t)(-1)\, dt = -\frac{1}{2}$$

$$\int_3^4 Re(\gamma)\gamma' = \int_3^4 0\, dt = 0$$

$$\Rightarrow \int_\gamma x\, dz = i \, .$$