ELLIPTIC FUNCTIONS (WEEK 13)

3. THE WEIERSTRASS $\wp$-FUNCTION

Following Weierstrass, we now create our first example of an elliptic function. The simplest examples will have order 2 (the smallest possible order) and necessarily have either a single double pole with residue zero, or two simple poles with opposite residues. Our example will have a double pole with residue zero.

We begin with a list of *desiderata* and their necessary implications. We want $\wp = \wp(\ ; \omega_1, \omega_2)$ to be elliptic with a double pole at 0 and periods $\omega_1, \omega_2 \in \mathbb{C}^\times$ such that $\omega_2/\omega_1 \notin \mathbb{R}$. Thus the leading term in the Laurent series of $\wp$ may as well be z^{-2}. Now $\wp(z) - \wp(-z)$ has the same periods and no singularity, hence is constant. Furthermore $\wp(\omega_1/2) - \wp(-\omega_1/2 = 0)$ by ω_1-periodicity of $\wp$, so $\wp(z) - \wp(-z) = 0$ for all z. We conclude that $\wp$ is an even function.

Addition of a constant is inconsequential, so let's demand that $\wp$'s constant term is 0. Thus we are on the hunt for a function of the form

$$\wp(z) = z^{-2} + a_1 z^2 + a_2 z^4 + a_3 z^6 + \cdots$$

with periods ω_1, ω_2.

Let $L = \mathbb{Z}\omega_1 + \omega_2$. We aim to show that

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L \smallsetminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

This is a reasonable formula to guess: we get poles of order 2 at all points in the period lattice, and $1/\omega^2$ is subtracted (making the summands roughly z/ω^3) to guarantee uniform convergence on compact sets. It is not obviously L-periodic (because of the $-1/\omega^2$ term), but you can show that

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z-\omega)^3}.$$

This function is clearly L-periodic, and you will combine this with evenness of $\wp$ to prove that $\wp$ has periods ω_1, ω_2 in a homework problem.

Having built up our desired properties, we will make one final definition and then state an omnibus theorem summarizing the properties of $\wp$.

Definition 3.1. The k-th *Eisenstein series* of a lattice L is

$$G_k = G_k(L) = \sum_{\omega \in L \smallsetminus \{0\}} \frac{1}{\omega^k}.$$

Remark 3.2. If k is odd, $G_k = 0$.

Theorem 3.3. *Let $\wp$ be the Weierstrass function with respect to a lattice L.*

(a) The Laurent expansion of $\wp$, valid for $0 < |z| < \min\{|\omega| \mid 0 \neq \omega \in L\}$, is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n}.$$

(b) The functions $\wp$ and $\wp'$ satisfy the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \tag{1}$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$.

(c) If $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, let $\omega_3 = \omega_1 + \omega_2$ and set $e_i = \wp(\omega_i/2)$ for $i = 1, 2, 3$. Then (1) is equivalent to

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \tag{2}$$

and the e_i are distinct.

Some interpretation of (b) and (c) is in order. By (1), we know that the pair $(\wp(z), \wp'(z))$ satisfies the equation

$$y^2 = 4x^3 - g_2 x - g_3$$

for $z \in \mathbb{C}$. It is in fact the case that the assignment

$$\begin{aligned} \mathbb{C}/L &\longrightarrow \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\}^* \\ z + L &\longmapsto (\wp(z), \wp'(z)) \end{aligned}$$

is a bijection (where the * indicates adding a point at ∞, and $L \mapsto \infty$). The object on the right is an algebraic geometer's notion of an elliptic curve, and this bijection explains the duplication of terminology.

Furthermore, (2) says that the right-hand side has roots e_1, e_2, e_3, giving the equivalent equation

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3).$$

Since the e_i are distinct, we call this equation *nonsingular*.

Proof of Theorem 3.3 (sketch). For (a), note that for $|z| < |\omega|$, the summand

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{(1 - z/\omega)^2} - 1\right) = \frac{1}{\omega^2}\sum_{n=1}^{\infty}(n+1)\frac{z^n}{\omega^n}$$

where the last equality follows from squaring the geometric series. Thus the summand is equal to $2z/\omega^3 + 3z^2/\omega^4 + \cdots$. Reordering the summations gives the desired identity.

For (b), compare the Laurent series in question. We have

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + O(z^6)$$

and

$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + O(z^5).$$

By some algebra, both $(\wp'(z))^2$ and $4\wp(z)^3 - g_2\wp(z) - g_3$ are of the form

$$\frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + O(z^2).$$

It follows that $(\wp'(z))^2 - (4\wp(z)^3 - g_2\wp(z) - g_3)$ is analytic and elliptic, hence constant. Since the difference is $O(z^2)$, it is also equal to 0.

For (c), recall that $\wp'$ is odd, and suppose that z is a point of order 2 in $\mathbb{C}/L$. Then $z \equiv -z \pmod{L}$, and $\wp'(z) = \wp'(-z) = -\wp'(z)$, whence $\wp'(z) = 0$. The order 2 points in $\mathbb{C}/L$ are exactly $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$, and (2) follows from (b). It remains to show that the e_i are distinct, but this follows because each is a double value of $\wp$ (since $\wp' = 0$ at the corresponding z-values) and $\wp$ has order 2. □

4. THE DISCRIMINANT AND j-FUNCTION

For $\tau \in \mathfrak{h} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, set $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$, the lattice with basis $(\tau, 1)$. We can turn the Eisenstein series into functions of the variable $\tau \in \mathfrak{h}$ by setting

$$G_k(\tau) = G_k(L_\tau).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned} G_k(\tau) &= G_k(L_\tau) \\ &= G_k(\gamma L_\tau) \\ &= G_k(\mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d)) \\ &= G_k((c\tau + d)\left(\frac{a\tau + b}{c\tau + d}\mathbb{Z} + \mathbb{Z}\right)) \\ &= (c\tau + d)^{-k} G_k(L_{\gamma\tau}) \\ &= (c\tau + d)^{-k} G_k(\gamma\tau). \end{aligned}$$

Here the second to last equality follows from the elementary observation that $G_k(mL) = m^{-k} G_k(L)$. Summarizing, we get

$$G_k(\gamma\tau) = (c\tau + d)^k G_k(\tau)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{h}$.

We now define the *discriminant function*

$$\begin{aligned} \Delta : \mathfrak{h} &\longrightarrow \mathbb{C} \\ \tau &\longmapsto g_2(\tau)^3 - 27 g_3(\tau)^2 \end{aligned}$$

which satisfies the transformation law

$$\Delta(\gamma\tau) = (c\tau + d)^{12} \Delta(\tau).$$

This permits the definition of *Klein's j-function*,

$$\begin{aligned} j : \mathfrak{h} &\longrightarrow \mathbb{C} \\ \tau &\longmapsto 1728 \frac{g_2(\tau)^3}{\Delta(t)} \end{aligned}$$

which is $\mathrm{SL}_2(\mathbb{Z})$-equivariant:

$$j(\gamma\tau) = j(\tau).$$

In fact, j is a holomorphic isomorphism between $X = \mathfrak{h}^* / \mathrm{SL}_2(\mathbb{Z})$ and the Riemann sphere (where $j(\infty) = \infty$). The space X is the *moduli space* of elliptic curves, and j specifies its topology and complex structure.

5. FIELDS OF MEROMORPHIC FUNCTIONS

A *Riemann surface* is a space in which every point admits an open neighborhood conformally equivalent to an open subset of $\mathbb{C}$. We have been working with three primary examples: open subsetes of $\mathbb{C}$, S^2, and $\mathbb{C}/L$. A more exotic example is the modular surface $\mathfrak{h}^* / \mathrm{SL}_2(\mathbb{Z})$.

One way to probe a Riemann surface is to understand its functions. Presently, we will concern ourselves with meromorphic functions on a Riemann surface X. These are the analytic functions $X \to S^2$ which are not constant with value ∞. As such, a function like $z \mapsto e^z/z$ is meromorphic on $\mathbb{C}$ but not on S^2. (It has an essential singularity at ∞.) We may pointwise add, subtract, multiply, and divide meromorphic functions on X (with some care, *i.e.*, limits, in cases like $0 \cdot \infty$), and this gives the set $K(X)$ of meromorphic functions on X the structure of a field. In general, functions on compact Riemann surfaces tend to be much simpler than on non-compact surfaces, and we will currently describe the meromorphic functions on S^2 and $\mathbb{C}/L$.

5.1. **Functions on the Riemann sphere.** Meromorphic functions on $S^2 = \mathbb{C} \cup \{\infty\}$ are particularly nice. First suppose that $f : S^2 \to S^2$ restricts to a function $f : \mathbb{C} \to \mathbb{C}$. This is our old notion of an entire function with the additional restriction that f has a nonessential singularity at ∞. By methods similar to one of Exam 2's problems, we can show that such functions are polynomial.

Now suppose that $f : S^2 \to S^2$ is analytic and takes the value ∞ (*i.e.* has poles as a function on $\mathbb{C}$) at $z_1, \dots, z_n \in \mathbb{C}$. If these poles have orders $k_1, \dots, k_n$, respectively, then the function

$$g : S^2 \longrightarrow S^2$$
$$z \longmapsto g(z) \prod_{i=1}^{n} (z - z_i)^{k_i}$$

is entire when restricted to $\mathbb{C}$. Thus g is a polynomial function, and

$$f(z) = \frac{g(z)}{\prod_{i=1}^{n} (z - z_i)^{k_i}}.$$

This proves the following theorem.

Theorem 5.1. *The field of meromorphic functions on the Riemann sphere equals the field of rational functions in a single variable,* i.e.,

$$K(S^2) = \mathbb{C}(z) = \{p(z)/q(z) \mid p, q \text{ polynomials with coefficients in } \mathbb{C},\ q \neq 0\}.$$

5.2. **Functions on elliptic curves.** Fix a lattice $L = \mathbb{Z}\omega_1 + \omega_2$ and let $\wp = \wp(\ ; L)$ be the associated Weierstrass $\wp$-function. Miraculously, we only need to know $\wp$ in order to know all of the meromorphic functions on $\mathbb{C}/L$.

Theorem 5.2. *The field $K(\mathbb{C}/L)$ consists of rational functions in $\wp$ and $\wp'$,* i.e.,

$$K(\mathbb{C}/L) = \mathbb{C}(\wp, \wp') = \left\{ \frac{f(\wp, \wp')}{g(\wp, \wp')} \;\middle|\; f, g \text{ polynomials in two variable with coefficients in } \mathbb{C},\ g \neq 0 \right\}.$$

Furthermore,

$$\mathbb{C}(\wp, \wp') \cong \mathbb{C}(x, y)/(y^2 = 4x^3 - g_2 x - g_3) = \mathbb{C}(x)(\sqrt{4x^3 - g_2 x - g_3})$$

the field of rational functions in two variables x, y subject to the relation $y^2 = 4x^3 - g_2 x - g_3$ where $g_i = g_i(L)$.

First note that $\mathbb{C}(\wp, \wp')$ is clearly a subfield of $K(\mathbb{C}/L)$, and the relation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

of Theorem 3.3 implies the final isomorphism. What is surprising here is that *every* meromorphic function on $\mathbb{C}/L$ can be expressed in such a fashion, and that is what we will concern ourselves with in the following sketch.

Proof Sketch. We begin with a reduction step that will allow us to only consider the even elliptic functions. Suppose f is meromorphic on $\mathbb{C}/L$ and let

$$f_1(z) = \frac{f(z) + f(-z)}{2}, \quad f_2(z) = \frac{f(z) - f(-z)}{2\wp'(z)}.$$

Since $\wp'$ is odd, both of these functions are even, and $f = f_1 + \wp' \cdot f_2$. As such, it suffices to prove that the field of even meromorphic functions on $\mathbb{C}/L$ is $\mathbb{C}(\wp)$.

Suppose $f : \mathbb{C}/L \to S^2$ is meromorphic and even. Our strategy is to produce an even meromorphic function φ on $\mathbb{C}/L$ which is rational in $\wp$ and has the same order of vanishing as f at all points. It will then follow that f/φ is analytic and elliptic, and thus is constant, from which we conclude that $f = c\varphi$ is rational in $\wp$ as well.

Let $\nu_0(f)$ denote the order of vanishing of f near 0. The Laurent series of f about 0 takes the form

$$f(z) = \sum_{n \geq \nu_0(f)} a_n z^n$$

where all powers of n are even and thus $\nu_0(f)$ is even. Near $\omega_1/2$, we have a similar expansion

$$f(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n (z - \omega_1/2)^n.$$

Define $g(z) = f(z + \omega/2)$, which is also meromorphic on $\mathbb{C}/L$. This function is also even since

$$g(-z) = f(-z + \omega_1/2) = f(-z - \omega_1/2 + \omega_1) = f(-z - \omega_1/2) = g(z).$$

Thus $\nu_0(g)$ is even as well. Additionally, the Laurent expansion of g about 0 is

$$g(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n z^n$$

so $\nu_{\omega_1/2}(f)$ is even as well. Via similar arguments, $\nu_{\omega_2/2}(f)$ and $\nu_{(\omega_1+\omega_2)/2}(f)$ are even as well.

Let $\{\pm z_1, \ldots, \pm z_n\}$ be the set of congruence classes of zeros or poles of f not of the form $(\varepsilon_1\omega_1 + \varepsilon_2\omega_2)/2$ for $\varepsilon_i = 0$ or 1. (The latter classes are precisely those z for which $z = -z$ in $\mathbb{C}/L$.) Let $(\mathbb{C}/L)[2]$ denote these 2-torsion points. Define φ by the formula

$$\varphi(z) = \prod_{i=1}^{n} (\wp(z) - \wp(z_i))^{\nu_{z_i}(f)} \prod_{w \in (\mathbb{C}/L)[2]} (\wp(z) - \wp(w))^{\nu_w(f)/2}.$$

(We have seen that $\nu_w(f)$ is even, and this value is 0 when w is not a zero or pole of f, in which case the term does not contribute to the product.) Clearly, this is a rational function in $\wp$. Furthermore, φ has the same order of vanishing as f everywhere since $\wp$ takes the values in W to order 2 and takes all other values to order 1. Thus we have produced the desired φ and $f = c\varphi$ is rational in $\wp$ as well, completing the argument. $\square$