Linear Fractional Transformations

$$h(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

(If $ad-bc=0$, then $h(z) = \frac{b}{d}$ $\forall z$)

Regard $h: S^2 \longrightarrow S^2$
$\infty \longmapsto \frac{a}{c}$
$-\frac{d}{c} \longmapsto \infty$

Want to understand Aut(U), conformal automorphisms of $U \subseteq S^2$, esp. $U = S^2$.

Thm Every lin frac trans is a conf aut of S^2. Conversely, each conf aut of S^2 is either an affine trans $L(z) = az+b$ or a comp'n $L_1 \circ s \circ L_2$, L_1, L_2 affine, $s(z) = \frac{1}{z}$. Hence each conf aut of S^2 is a lin frac trans.

pf If $h(z) = \frac{az+b}{cz+d} = w$, then $z = \frac{-dw+b}{cw-a} =: g(w)$.

$$h \circ g(w) = \frac{(ad-bc)w}{ad-bc} = w \text{ since } ad-bc \neq 0.$$

(Also check $w = \infty, -a/c$, and $g \circ h = id$.)

For converse, first suppose $f(\infty) = \infty$. Then $q = s \circ f \circ s : z \longmapsto \frac{1}{f(1/z)}$ is conf and $q(0) = 0$. This must be a 0 of order 1 ($q' \neq 0$) so $f(1/z)$ has a pole of order 1 at 0 and is analytic and finite at every other pt of S^2 $\Longrightarrow$ f has a pole of order 1 at ∞ and is analytic and finite on $\mathbb{C}$. Thus $\frac{f(z)-f(0)}{z}$ is analytic and finite on S^2 $\Longrightarrow$ const on S^2 (Liouville). For const a, $f(z) = az + f(0)$, affine.

Suppose $f(\infty) = k \neq \infty$. Set $p(z) = \frac{1}{z-k}$. Then

$$p \circ f(z) = \frac{1}{f(z)-k} \quad \text{conf aut of } S^2 : \infty \mapsto \infty.$$

Thus $p \circ f(z) = az+b \implies f(z) = \frac{1}{az+b} + k = \frac{akz+bk+1}{az+b}$. $\square$

Lines and Circles

Circle in S^2 = circle in $\mathbb{C}$ or line in $\mathbb{C} \cup \{\infty\}$.

Thm Each lin frac trans takes circles in S^2 to circles in S^2.

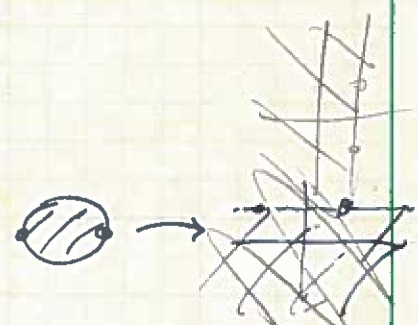
Pf Suffices to check affine trans and s. ✓

Lines/circles are solns to $a|z|^2 + \bar{w}z + w\bar{z} + b = 0$ for some $a, b \in \mathbb{R}$, $w \in \mathbb{C}$ (HW). Applying s to such a soln gives z satisfying $a|z|^{-2} + \bar{w}z^{-1} + w\bar{z}^{-1} + d = 0$

$$\underset{\cdot z\bar{z}}{\implies} a + \bar{w}z + w\bar{z} + d|z|^2 = 0 \quad ✓$$

e.g. $h(z) = \frac{2z}{z-i}$

$h(1) = 1+i$, $h(-1) = 1-i$, $h(i) = \infty$.



$h(\partial D, (0)) = \{z \mid \mathrm{Re}(z) = 1\} \cup \{\infty\}$

$h(0) = 0 \implies h(D,(0)) = \{z \mid \mathrm{Re}(z) < 1\}$.

3-Point Thm Given two ordered triples $(w_1, w_2, w_3), (z_1, z_2, z_3) \in (S^2)^{\times 3}$

$\exists!$ LFT h s.t. $h(w_j) = z_j$, $j = 1, 2, 3$.

Pf First take $(w_1, w_2, w_3) \in \mathbb{C}^3 \longrightarrow (0, 1, \infty)$ via $h(z) = \frac{(w_2 - w_3)(z - w_1)}{(w_2 - w_1)(z - w_3)}$

If $w_3 = \infty$, take $h(z) = \frac{z - w_1}{w_2 - w_1}$ to still map to $(0, 1, \infty)$.

Then if $(w_1, w_2, w_3) \xrightarrow{h_1} (0, 1, \infty) \xleftarrow{h_2} (z_1, z_2, z_3)$, get $h_2^{-1} \circ h_1$

as desired. Now suppose g is another such trans.

Then $f = h_2 \circ g \circ h_1^{-1} : (0,1,\infty) \to (0,1,\infty)$.

$$\begin{array}{l} \shortparallel \\ \dfrac{az+b}{cz+d} \end{array} \qquad \begin{aligned} &f(0)=0 \Rightarrow b=0 \\ &f(\infty)=\infty \Rightarrow c=0 \\ &f(1)=1 \Rightarrow \frac{a}{d}=1 \Rightarrow f = id. \end{aligned}$$

$$\Rightarrow g = h_2^{-1} \circ f \circ h_1 = h_2^{-1} \circ h_1 = h$$

$\square$

<u>Thm</u> For $D = D_1(0)$, $w \in D$, the LFT $h_w(z) = \dfrac{z-w}{1-\bar{w}z}$

satisfies $h_w(0) = -w$, $h_w(w) = 0$, $h_w(D) = D$, fixes ∂D pointwise.

The conformal automorphisms of D are the LFTs of the form $h(z) = u\,h_w(z)$ for some $|u|=1$, $|w|<1$.

Pf p.207. $\square$

Automorphisms

An LFT is an aut. of $\mathbb{C}$ when $\infty \mapsto \infty$ i.e. $c=0$ so

$f(z) = az+b$ is affine.

In fact, these are all bianalytic fns $\mathbb{C} \to \mathbb{C}$. (HW).

If $f(z) = az+b$, $g(z) = a'z+b'$, then

$$(f \circ g)(z) = aa'z + (ab'+b)$$
$$f^{-1}(z) = a^{-1}z - a^{-1}b \qquad \rightsquigarrow \text{group law.}$$

Easier: $f(z) = az+b \longleftrightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

whence matrix mult'n, inversion give the ops.

Defin $P := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{C}, a \neq 0 \right\} \cong \mathrm{Aut}(\mathbb{C})$

parabolic group

Defin Levi component $M = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{C}, a \neq 0 \right\}$ = (complex) dilations

unipotent radical $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{C} \right\}$ = translations

Prop $P = MN = NM$ and M normalizes N: $m^{-1}nm \in N$ $\forall m \in M$, $n \in N$.

Pf Check eq'ns. $\square$

Moral
- affine = dil'n $\circ$ trans = trans $\circ$ dilation
- dilation-then-trans-then-inv-dil'n is a trans.

$$\mathrm{Aut}(S^2) \cong \{\text{LFT's}\} \xleftarrow{\text{hom}} GL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} a,b,c,d \in \mathbb{C} \\ ad-bc \neq 0 \end{matrix} \right\}$$

Kernel $= \mathbb{C}^\times I = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}^\times \right\}$

So 1st iso thm $\Rightarrow$ $\mathrm{Aut}(S^2) \cong GL_2(\mathbb{C}) / \mathbb{C}^\times I =: PGL_2(\mathbb{C})$.

Define $SL_2(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}, ad-bc=1 \right\}$

Let $G = GL_2(\mathbb{C})$, $K = \mathbb{C}^\times I$, $H = SL_2(\mathbb{C})$. Note $H \cap K = \{\pm I\}$

Now $G = HK$ b/c
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\frac{1}{\sqrt{ad-bc}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in H} \cdot \underbrace{\begin{pmatrix} \sqrt{ad-bc} & 0 \\ 0 & \sqrt{ad-bc} \end{pmatrix}}_{\in K}$$

Thus $G/K = HK/K \cong H/H\cap K = SL_2(\mathbb{C})/\{\pm I\} =: PSL_2(\mathbb{C})$.

Hence $\boxed{\mathrm{Aut}(S^2) \cong PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})}$.

__The rotation subgp__

$\mathrm{Rot}(\text{unit sphere in } \mathbb{R}^3) \cong SO_3(\mathbb{R}) = \left\{ m \in M_{3\times 3}(\mathbb{R}) \mid \det m = 1, m^T m = I \right\}$

$\cong \mathrm{Rot}(S^2)$

$SU_2(\mathbb{C}) := \left\{ m \in M_{2\times 2}(\mathbb{C}) \mid \bar{m}^T m = I, \det m = 1 \right\}$

$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a,b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} \leq SL_2\mathbb{C}$

$= S^3$ with group structure!

Fact $PSU_2\mathbb{C} \cong \mathrm{Rot}(S^2)$
$\quad \| $
$SU_2\mathbb{C}/\pm I$

Pf $SU_2\mathbb{C} \to SO_3\mathbb{R}$
surj w/ ker $\pm I$.

<u>Disc</u> $D = D_1(0) \ni w$

$$h_w(z) \longleftrightarrow \begin{pmatrix} 1 & -w \\ -\bar{w} & 1 \end{pmatrix}$$

$$\mathrm{Aut}\, D \cong PSU_{1,1}(\mathbb{C})$$

$$SU_{1,1}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a,b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\}$$

<u>Upper half plane</u>

$$H = \{ x+iy \mid y>0 \} \xrightarrow[\cong]{t} D$$

$$z \longmapsto \frac{z-i}{-iz+1}$$

so $\mathrm{Aut}(H) = t^{-1}\mathrm{Aut}(D)\, t$

and $\frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} SU_{1,1}(\mathbb{C}) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = SL_2\mathbb{R}$

so $\mathrm{Aut}(H) \cong PSL_2(\mathbb{R})$.

Aut's of H fixing $i \longleftrightarrow SO_2\mathbb{R}$

and $SL_2\mathbb{R} = PK$ for $P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a,b,d \in \mathbb{R},\ ad = 1 \right\}$

G a locally compact Hausdorff top gp acting trans on a Hausdorff space X. For some $x \in X$ set G_x = isotropy subgp of x in G. Then $X \cong G/G_x$.

e.g. $S^2 \cong SO_3(\mathbb{R})/SO_2(\mathbb{R}) \cong SL_2(\mathbb{C})/P$

$H \cong SL_2\mathbb{R}/SO_2\mathbb{R}$

Riemann Mapping Theorem

Normal Families:

Defn $U \subseteq \mathbb{C}$ open. A collection $\mathcal{F}$ of analytic fns $U \to \mathbb{C}$ is a normal family if each sequence in $\mathcal{F}$ either converges uniformly to ∞ on each compact subset of U or has a subseq which converges uniformly on each compact subset of U to an analytic fn.

A collection $\mathcal{F}$ of analytic fns $U \to \mathbb{C}$ is uniformly bounded if $\exists M \geq 0$ s.t. $|f(z)| \leq M$ $\forall z \in U, f \in \mathcal{F}$.

Thm [Montel] If $\mathcal{F}$ is uniformly bounded, then it is normal.

Pf Let $\{f_n\}$ be a unif bdd sequence of fns on U. Show this has subseq that converges unif on each cpt subset of U.

First enumerate pts of U w/ rat'l coords: $z_1, z_2, \ldots$

Choose $M > 0$ s.t. $|f(z)| \leq M$ $\forall z \in U, f \in \mathcal{F}$. Then $\{f_n(z_1)\}$ is a seq of cpx #s bdd in modulus by $M \Rightarrow$ has conv subseq $\{f_{11}(z_1), f_{12}(z_1), \ldots\}$. Now $\{f_{1n}(z_2)\}$ bdd so has conv subseq $\rightsquigarrow \{f_{2n}\}$. Continuing inductively get $\{f_{kn}\}$ with each seq a subseq of the preceding one, and $\{f_{kn}(z_j)\}_n$ conv for $j \leq k$. Then $\{f_{nn}\}$ is a subseq converging at every z_j.

Set $g_n = f_{nn}$. Now show $\{g_n\}$ conv unif on cpt $\subseteq U$.

For $w \in U$ choose $r > 0$ s.t. $\bar{D}_{2r}(w) \subseteq U$. For $z \in \bar{D}_r(w)$

$$\bar{D}_r(z) \subseteq \bar{D}_{2r}(w) \subseteq U.$$

By Cauchy's estimates, each $f \in \mathcal{F}$ satisfies $|f'(z)| \le \frac{M}{r}$.

For $z, z' \in \bar{D}_r(w)$, have $|f(z) - f(z')| = \left|\int_z^{z'} f'(\lambda)\, d\lambda\right| \le \frac{M}{r}|z - z'|$

Given $\varepsilon > 0$ choose $\delta = \frac{r\varepsilon}{3M}$. If $z \in D_r(w)$, take z_j w/ rational coeffs st. $|z - z_j| < \delta$.

independent of $f \in \mathcal{F}$ so hold $\forall g_n$.

$$|g_n(z) - g_n(z_j)| < \frac{M}{r}\frac{r\varepsilon}{3M} = \frac{\varepsilon}{3}.$$

Next choose N st. $|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}$ for $n, m \ge N$. (can b/c $g_n(z_j)$ conv $\Rightarrow$ Cauchy) Then

$$|g_n(z) - g_m(z)| \le |g_n(z) - g_n(z_j)| + |g_m(z_j) - g_m(z)| < \varepsilon$$

if $n, m \ge N$. Hence g_n unif Cauchy on $D_r(w) \Rightarrow$ unif conv on $D_r(w)$.

For $K \subseteq U$ cpt, $\{D_r(w) \mid w \in K, D_{2r}(w) \subseteq U\}$ cover K. Take finite subcover. Since $\{g_n\}$ unif conv on these finitely many discs, unif conv on K. $\square$.

Riemann Mapping Thm

Take $U \ne \emptyset \subsetneq \mathbb{C}$ open, conn'd, st. every non-vanishing analytic fn on U has an analytic square root.

Fix $z_0 \in U$ and let $\mathcal{F} = \{$inj conf fns $U \xrightarrow{f} D = D_1(0)$ & $f(z_0) = 0\}$

WTS $\exists f \in \mathcal{F}$ that surjects onto D.

Lemma $\mathcal{F}$ is nonempty.

Pf Take $\lambda \in \mathbb{C} \setminus U$, set $f(z) = z - \lambda$ which is non-vanishing on U, hence has a $\sqrt{f} = g$. Since f inj, non-van, so is g.

By Open Mapping Thm, $g(\mathcal{U}) \supseteq \bar{D}_r(w_0)$, $0<r<\infty$.

Note: $g^2 = f$ so $\cancel{f(z)}\, g(z)^2 = (-g(z))^2 \Rightarrow$ if $w \in \text{im}(g)$ then $-w \notin \text{im}(g)$ (o/w f not inj!) Hence $\bar{D}_r(-w_0) \cap g(\mathcal{U}) = \emptyset$.

I.e. $|g(z)+w_0| > r$ $\forall z \in \mathcal{U}$. Thus $p(z) = \dfrac{r}{g(z)+w_0}$ is inj, conformal $\mathcal{U} \to D$. If $p(z_0) = w$, compose w/ h_w to get $h_w \circ p : \mathcal{U} \to D$ inj, conf taking $z_0 \mapsto 0$. $\square$

Note $p(z) = \dfrac{r}{\sqrt{z-\lambda} + w_0}$

Lemma $\mathcal{U}, z_0, \mathcal{F}$ as above. If $f \in \mathcal{F}$ and $f(\mathcal{U}) \subsetneq D$, then $\exists g \in \mathcal{F}$ with $|g'(z_0)| > |f'(z_0)|$.

Pf Take $w \in D \sim f(\mathcal{U})$. Then $h_w \circ f(z) \neq 0$ $\forall z \in \mathcal{U}$, so $h_w \circ f$ has an analytic $\sqrt{F} = q$. If $\lambda^2 = w$, $q^2 = h_w \circ f$ and $q(z_0) = \lambda$, then

$$q'(z_0) = \frac{h_w'(0)}{2q(z_0)} f'(z_0) = \cdots = \frac{(1-|\lambda|^4)}{2\lambda} f'(z_0).$$

Then $g = h_\lambda \circ q \in \mathcal{F}$ and

$$g'(z_0) = h_\lambda'(\lambda)\, q'(z_0) = \cdots = \frac{(1+|\lambda|^2)}{2\lambda} f'(z_0)$$

Now $0 < (1-|\lambda|)^2 = 1 - 2|\lambda| + |\lambda|^2 \Rightarrow 2|\lambda| < 1+|\lambda|^2$.

Also $|f'(z_0)| > 0$, so $|g'(z_0)| > |f'(z_0)|$. $\square$

Thm For $\mathcal{U}$ as above, there is a conf equiv $\mathcal{U} \to D$.

Pf For $z_0 \in \mathcal{U}$, $\mathcal{F}$ as before, know $\mathcal{F} \neq \emptyset$. Set $m = \sup\{|f'(z_0)| \mid f \in \mathcal{F}\}$

By prev lemma, existence of $h \in \mathcal{F}$ with $|h'(z_0)| = m$ implies $h(\mathcal{U}) = D$.

Choose seq $\{f_n\}$ in $\mathcal{F}$ s.t. $\lim_{n\to\infty} |f_n'(z_0)| = m$.

Since $\mathcal{F}$ is unif bdd (by 1) it is normal, whence there is a subseq of $\{f_n\}$ converging unif on cpt $K \subseteq \mathcal{U}$ to h.
Get $|h'(z_0)| = m$ by Cauchy's Estimates.*
Since $m \neq 0$, h is not const. Inj of $f_n \Rightarrow$ inj of h.**
Since $f_n(z_0) = 0 \ \forall n$, $h(z_0) = 0$. Thus $h \in \mathcal{F} \Rightarrow h$ conf equiv $\mathcal{U} \longrightarrow D$. $\square$

<u>Cor</u> For $\mathcal{U}$ as above, $\mathcal{U}$ is simply conn'd.
Every $\mathcal{U} \subseteq \mathbb{C}$ open, simply conn'd is conf equiv to D.