Summing Infinite Series

Hope: Compute $\sum_{n=-\infty}^{\infty} f(n)$, f analytic with isolated sings on $\mathbb{C}$

Idea: · Find g with simple pole with residue 1 at each $n \in \mathbb{Z}$.

· Then fg has a simple pole with residue $f(n)$ at $n \in \mathbb{Z}$.

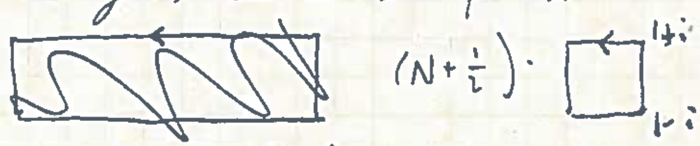
· Find $\{\gamma_N\}_N$ simple closed paths s.t. sings of fg contained in γ_N for $N \gg 0$ and s.t. $\int_{\gamma_N} fg \longrightarrow 0$ as $N \to \infty$.

· Then, by Residue Thm, $\sum \operatorname{Res}(fg, z_i) = 0$.

If f has only fin many sings, none of which are integers, then $\sum f(n) = -\sum_{\substack{w_i \text{ sing} \\ \text{of } f}} \operatorname{Res}(fg, w_i)$.

Good choice of g is $g(z) = \pi \cot(\pi z)$, which has a simple pole with residue 1 at each integer, and no other poles.

For $N \in \mathbb{Z}^+$, take γ_N:



These capture all sings and g is bdd on γ_N for $N \gg 0$:

<u>Lemma</u> $\exists R > 0$ s.t. $|\cot(\pi z)| \le 2$ on $\gamma_N(I)$ for $N \ge R$.

<u>Pf</u> $$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i\,\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}}$$

$$= i\,\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = i\,\frac{e^{2\pi i x} e^{-2\pi y} + 1}{e^{2\pi i x} e^{-2\pi y} - 1} \quad \text{for } z = x + iy.$$

Thus $|\cot(\pi z)| = \left| \dfrac{e^{2\pi i x} e^{-2\pi y} + 1}{e^{2\pi i x} e^{-2\pi y} - 1} \right|$. If $x = N + \frac{1}{2}$, $e^{2\pi i x} = -1$

$\Rightarrow |\cot(\pi z)| = \left| \dfrac{1 - e^{-2\pi y}}{1 + e^{-2\pi y}} \right| \le 1$ on vertical sides.

On horizontals, max for $e^{2\pi i x} = 1$ i.e. $x \in \mathbb{Z}$.

Thus $\max \left| \frac{e^{\pm(2N+1)\pi}+1}{e^{\pm(2N+1)\pi}-1} \right| \xrightarrow[N\to\infty]{} 1 \Rightarrow \exists N$ s.t. <2 for $N > M$. $\square$

<u>Lemma</u> With γ_N as above, $\int_{\gamma_N} \frac{\pi \cot \pi z}{z} = 0$ for each $N \in \mathbb{Z}^+$

<u>Pf</u> $\int_{\gamma_N} \underbrace{\frac{\pi \cot(\pi z)}{z}}_{h(z)} dz = 2\pi i \sum_{-N}^{N} \operatorname{Res}(h, n)$

$\operatorname{Res}(h,0) = 0$ since h even (HW)

$\underset{\substack{\neq \\ 0}}{\operatorname{Res}(h,n)} = \frac{\pi \cos(\pi n)/n}{\pi \cos(\pi n)} = \frac{1}{n}$ odd $\Rightarrow \int_{\gamma_N} = 0$. $\square$

$\sim$ simple pole

<u>Thm</u> Suppose f is analytic on $\mathbb{C}$ except at $E = \{z_1, \dots, z_m\}$ iso sings. Suppose $\exists R, M > 0$ s.t. $|f(z)| \underset{(*)}{\leq} \frac{M}{|z|}$ for $|z| \geq R$.

Then $\lim_{N \to \infty} \sum_{n \in [-N,N] \cap \mathbb{Z} \setminus E} f(n) = - \sum_{j=1}^{m} \operatorname{Res}(\pi f(z) \cot(\pi z), z_j)$.

<u>Pf</u> The sings of $\pi f(z) \cot(\pi z)$ are at $\mathbb{Z} \cup E$. For $n \in \mathbb{Z} \setminus E$, Res $= f(n)$ since $\pi \cot(\pi z)$ simple pole w/ res 1 at n, f analytic at n. For $N \gg 0$ s.t. E inside γ_N,

$$\underbrace{\frac{1}{2\pi i} \int_{\gamma_N} \pi f(z) \cot(\pi z)\, dz}_{} = \sum_{n \in [-N,N] \cap \mathbb{Z} \setminus E} f(n) + \sum_{j=1}^{m} \operatorname{Res}(\pi f(z) \cot(\pi z), z_j)$$

Thus suffices to show $\to 0$. as $N \to \infty$.

Choose $R, M > 0$ s.t. $(*)$ holds, $|\cot(\pi z)| \leq 2$ on γ_N for $N \geq R$, and $|z_j| \leq R$, $j = 1, \dots, m$. Then f is analytic in

$$A = \{z \mid R < |z| < \infty\}$$

and f vanishes at ∞.

Thus f has a Laurent exp'n of the form

$$f(z) = \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots \quad \text{on } A.$$

Since $\int_{\gamma_N} \frac{\pi \cot(\pi z)}{z} = 0$, get

$$\frac{1}{2\pi i}\int_{\gamma_N} \pi f(z)\cot(\pi z)\,dz = \frac{1}{2\pi i}\int_{\gamma_N} \pi\left(f(z) - \frac{c_{-1}}{z}\right)\cot(\pi z)\,dz$$

get smart

Now $f(z) - \frac{c_{-1}}{z} = \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} + \cdots = \frac{q(z)}{z^2}$ in A where q is analytic in A with $q(z) \longrightarrow c_{-2}$ as $z \to \infty$.

Thus for some $R_1 > R$, $|q(z)| < M_1$ on $\mathbb{C} \setminus D_{R_1}(0)$.

and integrand modulus bdd by $2MM_1/|z|^2$.

Since $|z| > N$ on γ_N, length bd gives

$$\left|\frac{1}{2\pi i}\int_{\gamma_N} \pi f(z)\cot(\pi z)\,dz\right| \le \frac{2MM_1(8N+4)}{N^2} \xrightarrow[N\to\infty]{} 0. \quad \square$$

<u>Thm</u> $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

<u>Pf</u> Preceding Thm with $f(z) = \frac{1}{z^2}$ gives

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = -\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2}, 0\right)$$

Have seen $\operatorname{Res}\left(\frac{\cot(z)}{z^2}, 0\right) = -\frac{1}{3} \implies \quad = \frac{\pi^2}{3}$.

Thus $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\,\frac{\pi^2}{3} = \frac{\pi^2}{6}$. $\square$

<u>Thm</u> $\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$ $\quad \left(\frac{\pi^2}{6}, \frac{\pi^4}{90}, \frac{\pi^6}{945}, \ldots\right)$

Conformal Mappings

Conformal = angle preserving

If $h = u + iv : U \to \mathbb{C}$, $\quad Jh := \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

If $\gamma = x + iy : [a,b] \to \mathbb{C}$ with $\gamma(t_0) = z_0$, then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is the tangent vector to γ at z_0

By chain rule, $(h \circ \gamma)'(t_0) = Jh(z_0) \cdot \gamma'(t_0)$

i.e. $(h \circ \gamma)' = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

<u>Defn</u> h is <u>conformal</u> at z_0 if angle b/w $Jh(z_0) \cdot a$ & $Jh(z_0) \cdot b$ $=$ angle(a, b) $\forall a, b \in \mathbb{R}^2 \setminus 0$, i.e.

$$\langle a, b \rangle = \langle Jh(z_0) \cdot a, Jh(z_0) \cdot b \rangle$$

If $h : U \to V$ conformal at all $z_0 \in U$ and surj, call h <u>conformal</u>.

<u>Thm</u> h is conformal at z_0 iff it has a nonzero cpx deriv at z_0.

<u>Pf</u> If h has a cpx deriv at z_0 then

$$Jh(z_0) = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$=$ dilation $\cdot$ rotation so conformal

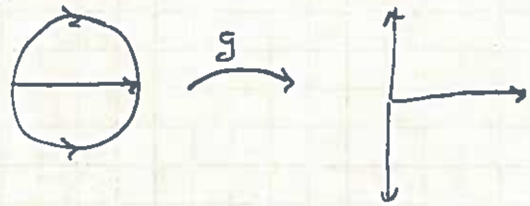
Converse: reading/exc. $\square$

<u>Thm</u> If h is an injective conformal map $U \to V$ then h has an inverse $h^{-1}: V \to U$ also conformal.

<u>Pf</u> Inverse mapping theorem. $\square$

e.g.
$$\begin{array}{ccc} z & \longmapsto & \frac{1+z}{1-z} \\ g : \mathbb{C}\setminus\{1\} & \longleftrightarrow & \mathbb{C}\setminus\{-1\} : g^{-1} \\ -\frac{1-w}{1+w} & \longleftarrow\!\shortmid & w \end{array}$$

When $|z|<1$, and $\mathrm{Re}(g(z))>0$ (and vice versa) so g restricts to a conformal equiv of $D_1(0) \longrightarrow$ right half plane



Thus $h : D_1(0) \longrightarrow H$ conf equiv
$$z \longmapsto i\,\frac{1+z}{1-z}$$

Now $H \longrightarrow$ 1st quadrant conv equiv
$$re^{i\theta} = z \longmapsto \sqrt{z} = \sqrt{r}\, e^{i\theta/2} \qquad -\pi < \theta < \pi$$

so $f : D_1(0) \longrightarrow$ 1st quad conf equiv
$$z \longmapsto \sqrt{i\,\frac{1+z}{1-z}}$$

e.g. $\mathrm{Log} : \{\mathrm{Re}\, z > 0\} \longrightarrow \{-\pi/2 < \mathrm{Im}(z) < \pi/2\}$ conf equiv

$$\begin{tikzcd} \{\mathrm{Re}\, z>0\} \arrow[r] & \{-\pi/2 < \mathrm{Im}(z) < \pi/2\} \arrow[d, "/\pi/2"] \\ D_1(0) \arrow[u, "g"] \arrow[r] & \{-1 < \mathrm{Im}(z) < 1\} \end{tikzcd}$$

$$z \longmapsto \frac{2}{\pi}\,\mathrm{Log}\left(\frac{1+z}{1-z}\right)$$

The Riemann Sphere

Idea $\mathbb{C}\cup\{\infty\}$ is a sphere

Coordinatize: write $S^2 := \mathbb{C}\cup\{\infty\}$.

$$\begin{tikzcd} \mathbb{C}-0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C} \arrow[d] & z \arrow[d, mapsto] \\ \mathbb{C} \arrow[r] & S^2 & z \end{tikzcd}$$

$$z \longmapsto \begin{cases} \frac{1}{z} = w & z \neq 0 \\ \infty & z = 0 \end{cases}$$

An open disc in S^2 centered at ∞ is $\{w \mid |w|<r\} = \{z \mid |z| > 1/r\}$

An open set in S^2 is a set U s.t. every pt in U is the center of some open disc contained in U.

A nbhd of ∞ = open set containing ∞

= $\{\infty\} \cup$ open subset of $\mathbb{C}$ containing the complement of a closed bdd disc centered at 0.

If $f: U \longrightarrow S^2$ for $U \subseteq S^2$ open.

$\lim_{z\to z_0} f(z) = L$ means $\forall$ nbhd W of L, there is a deleted nbhd V of z_0 s.t. $V \subseteq U$ and $f(V) \subseteq W$.

Analytic functions on S^2 "analytic at z_0" = defined and analytic in a nbhd of z_0.

Defn Say $f(z)$ is *analytic at ∞* if $f(1/w)$ is analytic at $w=0$ (i.e. defined and analytic in a deleted nbhd of 0 with removable sing at 0).

e.g. $f(z) = \dfrac{1-z}{1+z}$ is analytic at ∞ if we set $f(\infty) = -1$.

Indeed, $f(1/w) = \dfrac{1-1/w}{1+1/w} = \dfrac{w-1}{w+1}$ which is analytic at $w=0$.

Analytic functions $U \to S^2$.

For $U \subseteq S^2$ open, $f : U \to S^2$, call f analytic at z_0 with $f(z_0) = \infty$ iff $1/f$ is analytic at z_0.

Write $s : S^2 \to S^2$
$$z \longmapsto \begin{cases} 1/z & z \neq 0, \infty \\ \infty & z = 0 \\ 0 & z = \infty \end{cases}$$
so f is analytic at ∞ iff $f \circ s$ analytic at 0

f is analytic at z_0, $f(z_0) = \infty$ iff $s \circ f$ analytic at z_0.

<u>Thm</u> s is a conformal equivalence $S^2 \to S^2$. $\square$

Meromorphic fn f on $\mathbb{C}$ is said to have a pole of order n at ∞ if $f(1/w)$ has a pole of order n at 0. (Allow $n = 0$.)

<u>Thm</u> Each meromorphic fn on $\mathbb{C}$ with a pole at ∞ defines an analytic fn $\tilde{f} : S^2 \to S^2$, and each analytic function $S^2 \to S^2$ arises in this way.

<u>Pf</u> Suppose f meromorphic on $\mathbb{C}$ with a pole at ∞.

Define $\tilde{f}(z) = \begin{cases} f(z) & z \text{ not a pole} \\ \infty & z \text{ a pole of pos order} \end{cases}$

(If f has a pole of order 0 at ∞, then $f(1/w)$ has rem sing at 0 and $\tilde{f}(\infty)$ is the value at $w = 0$ making $f(1/w)$ analytic.)

Check $\tilde{f}$ analytic everywhere.

Suppose $g : S^2 \to S^2$ analytic. Where $g(z_0) \neq \infty$, g is analytic as a cpx valued fn. At $z_0 \in \mathbb{C}$ where $g(z_0) = \infty$, $\frac{1}{g(z)}$ is analytic hence $g(z)$ has a pole. $\square$

Complex Projective Space

$$\mathbb{CP}^1 = \left(\mathbb{C}^2 \setminus \{(0,0)\}\right) \Big/ (z,w) \sim (\lambda z, \lambda w) \text{ for } \lambda \in \mathbb{C} \setminus \{0\}$$

$$= \left\{ [z:w] \;\middle|\; (z,w) \in \mathbb{C}^2 \setminus \{(0,0)\} \right\} \text{ where } [z:w] = \{\lambda(z,w) \mid \lambda \in$$

$$\begin{array}{ccccc} \phi : \mathbb{CP}^1 \setminus \{[1:0]\} & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{CP}^1 \setminus \{[0:1]\} : \psi \\ [z:w] & \longmapsto & \frac{z}{w} & & \\ & & \frac{w}{z} & \longleftarrow\!\shortmid & [z:w] \end{array}$$

Again represents $\mathbb{CP}^1$ as two copies of $\mathbb{C}$ glued along $\mathbb{C} \setminus 0$.
so $\mathbb{CP}^1 \cong S^2$.

Stereographic Projection