Complex Analysis: The study of

$$\underbrace{\text{holomorphic}}_{\text{complex differentiable on } \Omega} \Big/ \underbrace{\text{meromorphic}}_{\text{holomorphic away from a discrete set of isolated points in } \Omega} \text{ functions } \quad f: \underset{\text{open subset of } \mathbb{C}}{\Omega} \longrightarrow \underset{\text{complex numbers}}{\mathbb{C}}$$

$$f \text{ is holomorphic at } z \in \Omega \text{ if } \underbrace{\lim_{h \to 0}}_{} \frac{f(z+h)-f(z)}{h} \text{ exists}$$

- complex limit
- $h \to 0$ in all possible directions!

Applications:

- analytic continuation (Riemann hypothesis, prime number theorem)
- algebraic geometry (elliptic curves)
- conformal mappings (conformal transforms of harmonic functions are harmonic — fields determined by potentials)

...

To understand complex functions, we'd better start with the

Complex Numbers

$\mathbb{C}$ starts life as the 2-dimensional real vector space spanned by $1, i$, so an element of $\mathbb{C}$ is of the form $a \cdot 1 + b \cdot i =: a+bi$ where $a, b \in \mathbb{R}$, and

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

for $c, d \in \mathbb{R}$.

Multiplication of basis elements:

$\cdot$	1	i
1	1	i
i	i	$? = i^2$

We know the punchline:

$$\boxed{i^2 = -1}.$$

Thus $(a+bi)\cdot(c+di) = ac + adi + bci + bdi^2$

$$= (ac-bd) + (ad+bc)i .$$

Note ~~$bi = (b+0i)\cdot(0+i) =$~~

$$ib = (0+i)(b+0i) = 0 + bi = bi$$

so $a+bi = a+ib$.

Thm $\mathbb{C} = \{a+bi \mid a,b\in\mathbb{R}\}$ is a field:

- commutative, associative addition
- commutative, associative multiplication
- additive, multiplicative units
- additive inverses
- multiplicative inverses in $\mathbb{C} - \{0\}$.
- multiplication distributes over addition.

Pf Review your 112 notes! Trickiest bit: for $z = a+bi$,

$$z^{-1} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = \frac{\overline{z}}{z\overline{z}} \qquad \square$$

Defn If $z = a+bi$, then $\overline{z} = a-bi$ is its complex conjugate.

The modulus (or norm) of z is $|z| = \sqrt{a^2+b^2}$.

The real part of z is $\mathrm{Re}(z) = a = \frac{1}{2}(z+\overline{z})$.

The imaginary part of z is $\mathrm{Im}(z) = b = \frac{1}{2i}(z-\overline{z})$.

Facts If $z,w\in\mathbb{C}$, then

(a) $\overline{\overline{z}} = z$

(b) $z\overline{z} = |z|^2$

(c) $\overline{z+w} = \overline{z}+\overline{w}$

(d) $\overline{zw} = \overline{z}\,\overline{w}$

(e) $|zw| = |z||w|$ and $|\overline{z}| = |z|$

(f) $|z| \geq 0$ with equality iff $z=0$

(g) $|\mathrm{Re}(z\bar{w})| \le |z||w|$ [Cauchy-Schwarz]

(h) $|z+w| \le |z| + |w|$ [triangle]

Pf (g) Note $\overline{\bar{z}w} = z\bar{w}$ so $z\bar{w} + \bar{z}w = 2\mathrm{Re}(z\bar{w})$.

For $t \in \mathbb{R}$, $0 \le |zt+w|^2 = (zt+w)(\bar{z}t+\bar{w})$

$$= |z|^2t^2 + 2\,\mathrm{Re}(z\bar{w})\,t + |w|^2$$

so RHS has at most one root in $\mathbb{R}$

$$\Rightarrow \underbrace{4(\mathrm{Re}(z\bar{w}))^2 - 4|z|^2|w|^2}_{\text{part under } \sqrt{\ } \text{ in quadratic fmla}} \le 0 \qquad \square$$

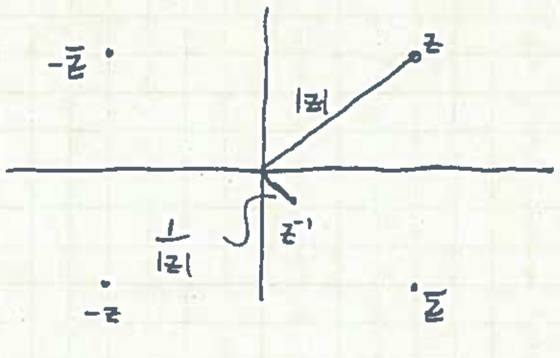
Pf (h) $|z+w|^2 = (z+w)(\bar{z}+\bar{w})$

$$= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2$$

$$\le |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2 \qquad \square$$

TPS · Draw $z, \bar{z}, -z, -\bar{z}, z^{-1}$

· Compute $\frac{1}{2+3i}$



Thm If $z = a+bi$, then $\max\{|a|, |b|\} \le |z| \le |a|+|b|$.

Convergence in $\mathbb{C}$

Defn A sequence $\{z_n\}$ of complex numbers converges to $w \in \mathbb{C}$ if $\forall\, \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |z_n - w| < \varepsilon$.

In this case, write $\lim z_n = w$ or $z_n \longrightarrow w$.

Note (a) $\lim z_n = w$ iff $\lim |z_n - w| = 0$

(b) For $\{a_n\}, \{b_n\}$ sequences of real numbers with $0 \leq a_n \leq b_n$, if $\lim b_n = 0$, then $\lim a_n = 0$. [squeeze principle]

$N = 6$ for this ε.

Thm A sequence $\{z_n\}$ of complex numbers converges to $w \in \mathbb{C}$ iff $\{Re(z_n)\}$ converges to $Re(w)$ and $\{Im(z_n)\}$ converges to $Im(w)$.

Pf $|Re(z_n) - Re(w)|, |Im(z_n) - Im(w)| \underset{Ⓐ}{\leq} |z_n - w| \underset{Ⓑ}{\leq} |Re(z_n) - Re(w)| + |Im(z_n) - Im(w)|$

Suppose $\lim z_n = w$. Then $\lim |z_n - w| = 0$, so $Re(z_n) \to Re(w)$ & $Im(z_n) \to Im(w)$ by Ⓐ.

Suppose $\lim Re(z_n) = Re(w)$ & $\lim Im(z_n) = Im(w)$.

Then $\lim z_n = w$ by Ⓑ. ☐

e.g. $\lim \left(\frac{1}{n} + \frac{n^2}{2n^2+1} i \right) = \frac{1}{2} i$.

Thm If $\lim z_n = z$ and $\lim w_n = w$, then $\lim (z_n + w_n) = z + w$ and $\lim (z_n w_n) = zw$. [Limits respect addition and mult'n.]

TPS Converse?

Pf Reading (Example 1.2.6). $\square$

Series of complex numbers

A series of complex numbers is a formal sum

$$\sum_{k=0}^{\infty} z_k = z_0 + z_1 + z_2 + \cdots$$

with $z_k \in \mathbb{C}$. Its n-th partial sum is $s_n = \sum_{k=0}^{n} z_k$.

$\sum_{k=0}^{\infty} z_k$ converges to w if $s_n \to w$.

Thm [Term test] If $\sum_{k=0}^{\infty} z_k$ converges, then $z_k \to 0$. $\square$

e.g. $\sum_{k=0}^{\infty} z^k$ converges to $\frac{1}{1-z}$ for $|z|<1$; diverges if $|z| \geq 1$.

Indeed, $s_n = \frac{1-z^{n+1}}{1-z}$ so $\left|s_n - \frac{1}{1-z}\right| = \frac{|z|^{n+1}}{|1-z|} \to 0$.

Defn $\sum_{k=0}^{\infty} z_k$ converges absolutely if $\sum_{k=0}^{\infty} |z_k|$ converges.

Thm If $\sum_{k=0}^{\infty} z_k$ converges absolutely, then it converges, and

$$\left|\sum_{k=0}^{\infty} z_k\right| \leq \sum_{k=0}^{\infty} |z_k|$$

Pf Reading [check on real, im parts; use Δ id]. $\square$

e.g. $\sum_{k=1}^{\infty} \frac{1}{k+k^2 i}$ converges:

$|k+k^2 i| \geq k^2$ so $\frac{1}{|k+k^2 i|} \leq \frac{1}{k^2}$. The p-series $\sum_{k=1}^{\infty} k^{-2}$ converges, so so does $\sum_{k=1}^{\infty} \frac{1}{|k+k^2 i|}$ by the comparison test. Thus $\sum_{k=1}^{\infty} \frac{1}{k+k^2 i}$ is absolutely convergent.

Power Series

A *complex power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $a_n, z_0 \in \mathbb{C}$, z a variable. We say this series is *centered* at z_0; it defines a function $\Omega \longrightarrow \mathbb{C}$ for $\Omega = \{z \in \mathbb{C} \mid \text{series converges at } z\}$.

Let $D_r(z_0) = \{z \in \mathbb{C} \mid |z-z_0| < r\}$

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z-z_0| \leq r\}.$$

From Math 112, know $\exists R \in \mathbb{R}_{>0} \cup \{\infty\}$ s.t.

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for $z \in D_R(z_0)$

diverges for $z \notin \bar{D}_R(z_0)$;

call R the *radius of convergence* of the power series.

TPS How do we find R?

A The ratio test on the absolute series!

e.g. What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{(3n)!\,z^n}{n!\,(2n)!}$?

Let $a_n = \frac{(3n)!\,z^n}{n!\,(2n)!}$. Then $\frac{|a_{n+1}|}{|a_n|} = \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+2)(n+3)}|z|$

$$= \frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}|z| \longrightarrow \frac{27}{4}|z|$$

Thus $\sum a_n$ is absolutely convergent for $|z| < \frac{4}{27}$, divergent for $|z| > \frac{4}{27}$, and $R = \frac{4}{27}$.

The Exponential Function

Defn $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ is the complex exponential function.

$$z \longmapsto e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note Converges on $\mathbb{C}$ by ratio test.

Thm $e^{z+w} = e^z e^w$

Pf The binomial theorem tells us $(z+w)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j}$.

Thus $$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} z^j w^{n-j}$$

$$= \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{z^j w^k}{j!k!}$$

$$= e^z e^w$$ (expanding product of power series). ☐

The last step is valid b/c absolutely convergent power series have product $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \sum_{j+k=n} a_j b_k$.

Now $e^{a+bi} = e^a e^{ib}$. We understand e^a from Math 111.

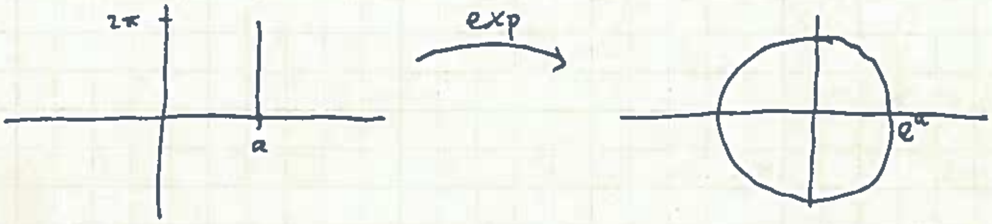
For e^{ib}, note $$e^{ib} = \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} \underbrace{=}_{\text{TPS Fmla for } i^n.} \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k+1}}{(2k+1)!}$$

$$\boxed{e^{ib} = \cos b + i \sin b}$$

Note e^{ib} is on the unit circle, b radians ccw from positive real axis.

Visualizing exp:

- exp is $2\pi i$ periodic: $e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1$
- So if we understand exp on $\{z \in \mathbb{C} \mid -\pi < \mathrm{Im}(z) \leq \pi\}$, then we understand exp on all of $\mathbb{C}$ (rolling up the plane).
- The segment $a + \overset{(-\pi,\pi]i}{\cancel{[0,2\pi)i}}$ becomes $e^a e^{\overset{(-\pi,\pi]i}{\cancel{[0,2\pi)i}}}$ = circle of radius e^a centered at 0.



Properties of exp:

Thm (a) $e^z \neq 0 \ \forall z \in \mathbb{C}$

(b) $|e^z| = e^{\mathrm{Re}(z)}$

(c) $|e^z| \leq e^{|z|}$

(d) $e^z = 1$ iff $z = 2\pi n i$ for some integer n.

TPS
- Derive the angle addition fmlae for cos, sin using $e^{i\theta} = \cos\theta + i\sin\theta$
- Create fmlae for $\cos(n\theta)$, $\sin(n\theta)$ for $n \in \mathbb{N}$.

Polar Form $\forall\, 0 \neq z \in \mathbb{C}\ \exists!\ r > 0,\ \theta \in \overset{(-\pi,\pi]}{\cancel{[0,2\pi)}}$ s.t $\boxed{z = re^{i\theta}}$.

For $z = 0$, take $r = 0$ and any θ.

Defn The argument of $z = re^{i\theta}$ is $\theta + 2\pi\mathbb{Z} = \{\theta + 2\pi n \mid n \in \mathbb{Z}\}$, denoted $\arg(z)$.

TPS Multiply + divide cpx #s in polar form.

<u>Thm</u> If $z = re^{i\theta} \neq 0$, then z has exactly n n-th roots,

$$r^{1/n} e^{i(\theta/n + 2\pi k/n)}, \quad k = 0, 1, \ldots, n-1.$$

<u>e.g.</u> The "n-th roots of $\underbrace{\text{unity}}_{\text{i.e. } 1}$" are $e^{2\pi k i/n}$, $k = 0, \ldots, n-1$.

5th roots of 1.

<u>Complex Logarithm</u>

$$z = re^{i\theta} = e^{\underset{\substack{|\\ \text{real natural}\\ \text{log}}}{\log r} + i\theta}$$ so we would like to define

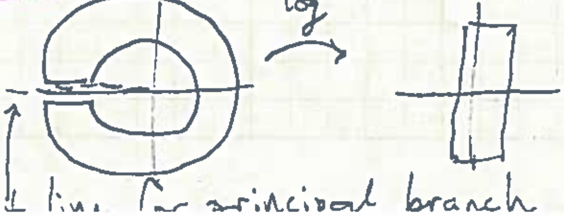
$$\log z = \log r + i\theta = \log|z| + i\underbrace{\arg(z)}_{\text{infinitely many values!}}$$

<u>Solution</u> Restrict $\theta = \arg(z)$ to lie in a half-open interval $I \subseteq \mathbb{R}$ of length 2π.

<u>Defn</u> Given a half-open interval I of length 2π, let $\arg_I z$, for $0 \neq z \in \mathbb{C}$, be the element of $\arg(z) \cap I$. Then the function $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $\log(z) = \log|z| + i \arg_I z$ is the <u>branch</u> of the log function defined by I. When $I = (-\pi, \pi]$, call this the <u>principal branch</u> of log.

<u>TPS</u> Let log be some branch of log.

- What is $\log(zw)$? $= \log(z) + \log(w) + 2\pi k i$ for some $k \in \mathbb{Z}$
- $\log(1)$?
- Is log continuous?



cut line for principal branch

Read Thm 1.4.8 for standard properties of log.

Other functions Given a branch of log, define

$$z^{1/n} = e^{(1/n)\log(z)} \text{ for } z \neq 0, \quad 0^{1/n} = 0.$$

In particular, $\sqrt{z} = e^{\frac{1}{2}\log(z)}$

◇? Discontinuities along negative real axis!