MATH 311: COMPLEX ANALYSIS
REVIEW PROBLEMS FOR EXAM 2

The second exam is a two-hour take-home exam which will be distributed on Monday, 1 April and is due Friday, 5 April at the start of class. You are permitted black scratch paper and one sheet of notes (front and back) but no other resources on the exam. The exam will focus on sections 2.5 through 4.3 (inclusive) in the textbook, including multiple versions of Cauchy's theorem and integral formula, Liouville's theorem, zeroes and singularities, the maximum modulus principle, power series, and Laurent series.

The following problems are good ones to practice with in preparation for the exam. I am happy to discuss them in office hours, via email, or on the Slack channel, but do not anticipate writing detailed solutions unless a particular topic needs extra review by multiple students. I also recommend reviewing your notes and graded homework assignments.

Problem 1. Let
$$f(z) = \frac{\pi z(1-z^2)}{\sin(\pi z)}.$$
(a) Identify all singularities of f in $\mathbb{C}$ and classify each as removable, a pole (of what order?), or essential.
(b) How do you know that $f(z)$ has a series expansion of the form $\sum_{k=-\infty}^{\infty} c_k z^k$? Find c_0, c_1, c_2.
(c) What is the largest open set on which the series in (b) converges?

Problem 2. Let $f(z) = e^{z^2}$. Determine $f^{(68)}(0)$ without computing $f^{(68)}(z)$, the 68th derivative of f.

Problem 3. Find the Laurent expansion of
$$f(z) = \frac{1}{z(z^2+1)}$$
that is valid for
(a) $0 < |z| < 1$
(b) $1 < |z|$.

Problem 4. The *Bernoulli numbers* B_n are defined to be the coefficients of the power series of $z/(e^z - 1)$:
$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$
(a) Determine the radius of convergence of this power series.
(b) Using the Cauchy integral formula and the contour $|z| = 1$, find an integral expression for B_n of the form
$$B_n = \int_0^{2\pi} g_n(\theta)\, d\theta$$
for suitable functions g_n.

Problem 5. Suppose that f is entire and that $\lim_{z\to\infty} f(z)/z = 0$. Prove that f is constant.

Problem 6. Suppose $f = u + iv$ is analytic on $U \subseteq \mathbb{C}$ with u, v real-valued. Prove that the level curves of u and v intersect orthogonally. (A *level curve* of u is a set $\{z \in \mathbb{C} \mid u(z) = c\}$ for some fixed c, and similarly for v.)