MATH 311: COMPLEX ANALYSIS
HOMEWORK DUE FRIDAY WEEK 7

Problem 1. Use power series to find a function f such that $f(0) = 1$ and $f'(x) = xf(x)$ for all x.

Problem 2. Prove the following complex version of l'Hôpital's rule: Let f, g be analytic, both having zeroes of order k at z_0. Then f/g has a removable singularity at z_0 and

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}.$$

Problem 3. We have seen that $1/(e^z - 1)$ has a Laurent series around $z = 0$ of the form

$$\frac{1}{e^z - 1} = \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \cdots .$$

(In particular, $1/(e^z - 1)$ has a simple pole at $z = 0$.) Determine b_1, a_0, a_1, and a_2. (You do not need to give a general expression for a_n in this problem.)

Problem 4. Find the residues of the following functions at the indicated points:
(a) $1/(z^2 - 1)$, $z = 1$
(b) $(e^z - 1)/z^2$, $z = 0$
(c) $(e^z - 1)/z$, $z = 0$

Problem 5. Find where the function $|e^z|$ attains its maximum value on $\overline{D}_1(0)$.

Problem 6. Show that $f(z) = (2z - 1)/(z - 2)$ is a bi-analytic map $\overline{D}_1(0) \to \overline{D}_1(0)$ which takes 0 to $1/2$.

Problem 7. Find a harmonic conjugate for $u(x, y) = 1/2 \log(x^2 + y^2)$ on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Problem 8. Consider the following directed line segments: γ_1 from -1 to $-1 + i$, γ_2 from 1 to $1 + i$, γ_3 from -1 to 0, γ_4 from 0 to 1, γ_5 from $-1 + i$ to i, γ_6 from i to $1 + i$, and γ_7 from 0 to i. Let $\Gamma = \gamma_1 + \gamma_2 - \gamma_3 + \gamma_4 + \gamma_5 - \gamma_6 - 2\gamma_7$.
(a) Show that Γ is a cycle.
(b) Find a sum of closed paths which is equivalent to Γ.
(c) Find $\mathrm{Ind}_\Gamma(z)$ for z in each component of $\mathbb{C} \setminus \Gamma(I)$.

Problem 9. Fix an integer n, let $\gamma(t) = e^{2\pi n i t}$, and let $\gamma_1(t) = 2e^{2\pi i t}$ for $t \in [0, 1]$. Show that the cycle $\gamma - n\gamma_1$ is homologous to 0 in $A = \{z \in \mathbb{C} \mid 0 < |z| < 3\}$.