MATH 311: COMPLEX ANALYSIS
HOMEWORK DUE FRIDAY WEEK 6

Problem 1. Use the power series expansion of $1/(1-z)$ about 0 to find the power series expansion of $1/(1-z)^2$ about 0.

Problem 2. Determine the raidus of convergence of the power series about $z_0 = 0$ for the function $1/\cos z$. (You do not need to determine the power series to solve this problem!)

Problem 3. Prove that if f is analytic on the disc $D_R(z_0)$ and $|f(z)| \leq M$ for $z \in D_R(z_0)$, then $|f'(z_0)| \leq M/R$. (Note that the hypothesis is different from that of Theorem 3.2.9 [Cauchy's Estimates].)

Problem 4. Use Morera's Theorem to show that if f is continuous on an open set U and analytic on $U \setminus E$, where E is either a point or a line segment, then f is actually analytic on all of U.

Problem 5. Prove that if an entire function f is not constant, then $f(\mathbb{C})$ is dense in $\mathbb{C}$. *Hint*: If $f(\mathbb{C})$ is not dense, then $f(\mathbb{C}) \cap D_r(z_0) = \varnothing$ for some $z_0 \in \mathbb{C}$ and $r > 0$.

Problem 6. Use Cauchy's estimate to prove that the only entire functions f satisfying $|f(z)| \leq A + B|z|^n$ for all $z \in \mathbb{C}$ are polynomials of degree at most n. (This completes the proof of Theorem 3.3.10 in the book.)

Problem 7. Fix a positive number K. Show that if f is an entire function and $|f(z)| \leq K|e^z|$ for all $z \in \mathbb{C}$, then $f(z) = Ce^z$ for some constant $C \in \mathbb{C}$.

Problem 8. Prove that a set E is a discrete subset of an open set U if and only if no sequence of distinct points in E converges to a point in U.

Problem 9. Show that if f is an analytic function with a zero of order k at z_0, then there is a neighborhood V of z_0 and an analytic function g on V such that $f = g^k$ on V and $g'(z_0) \neq 0$.

Problem 10. For the following functions, analyze each singularity. Is it removable? a pole? essential? If it is a pole, what is its order? If it is removable, what value should you give the function at that point to make it analytic.

(a) $f(z) = \dfrac{1}{z - z^3}$

(b) $g(z) = \dfrac{e^z - 1 - z}{z^2}$

(c) $h(z) = \dfrac{\mathrm{Log}(z)}{(1-z)^2}$ where Log is the principal branch of the logarithm