MATH 311: COMPLEX ANALYSIS
HOMEWORK DUE FRIDAY WEEK 3

Problem 1. Derive the Cauchy-Riemann equations in polar coordinates:

$$u_r = v_\theta / r$$
$$u_\theta = -r v_r$$

by using the change of variable formulas $x = r\cos\theta$, $y = r\sin\theta$ and the chain rule.

Problem 2. Verify that the function $\log|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$ and find a harmonic conjugate for it on the set $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Problem 3. Compute the following two Riemann integrals of complex-valued functions:
(a) $\int_0^\pi e^{it}\,dt$,
(b) $\int_0^1 \sin(it)\,dt$.

Problem 4. Fix $w \in \mathbb{C}$. Find (by direct computation) $\int_\gamma z^2\,dz$ if γ traces the straight line from 0 to w.

Problem 5. Prove (by direct computation) that $\int_\gamma p = 0$ where $\gamma : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(t) = e^{it}$ and p is any polynomial function. (This is a special case of Cauchy's Theorem, but you should not assume Cauchy's Theorem in your proof.)

Problem 6. Suppose $\Omega \subseteq \mathbb{C}$ is an open set and $f : \Omega \to \mathbb{C}$ is a function. A *primitive* for f on Ω is a function F that is analytic on Ω and such that $F' = f$. Suppose f is continuous on Ω with primitive F, and that γ is a smooth path[1] in Ω that begins at z_0 and ends at z_1. Prove that

$$\int_\gamma f = F(z_1) - F(z_0).$$

This is the fundamental theorem of calculus for contour integrals.

Problem 7. If $z_0, z_1 \in \mathbb{C}$ and γ is any smooth path which begins at z_0 and ends at z_1, compute $\int_\gamma z\,dz$. (You may use the result from Problem 6.)

[1]The result also holds for piecewise smooth paths, and you are invited to extend your proof to this setting.