MATH 311: COMPLEX ANALYSIS
HOMEWORK DUE WEDNESDAY WEEK 13

Note: Anyone may take advantage of a free extension to Friday if they would like.

Problem 1. For the following lattices, determine their canonical bases (ω_1, ω_2), $\tau = \omega_2/\omega_1$, and a matrix $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ taking the given basis to the canonical basis.

(a) $\mathbb{Z}(1+2i) + \mathbb{Z}(1+i)$

(b) $\mathbb{Z}(6+i\sqrt{3}) + \mathbb{Z}(3/2+i\sqrt{3}/2)$

Problem 2. Complete the proof of Theorem 2.13 from the Week 12 notes. In particular,

(a) Check that the residue theorem implies that the right-hand side of the first display on p.6 is

$$a_1 + \cdots + a_n - b_1 - \cdots - b_n.$$

(b) Prove the second displayed formula under the hypotheses of the theorem:

$$\frac{1}{2\pi i}\left(\int_a^{a+\omega_1} - \int_{a+\omega_2}^{a+\omega_1+\omega_2}\right)\frac{zf'(z)}{f(z)}\,dz = -\frac{\omega_2}{2\pi i}\int_a^{a+\omega_1}\frac{f'(z)}{f(z)}\,dz.$$

Problem 3. Let $L = \mathbb{Z} + \mathbb{Z}i$ denote the *Gaussian lattice* in $\mathbb{C}$, let $\wp = \wp(\ ;\mathbb{Z}+\mathbb{Z}i)$, and set $e_1 = \wp(1/2)$, $e_2 = \wp(i/2)$, $e_3 = \wp((1+i)/2)$.

(a) Use properties of $\wp$ to show that $\wp(iz) = -\wp(z)$.

(b) Use (a) to show that $e_1 = -e_2$ and $e_3 = 0$.

(c) What does $L = \overline{L}$ tell you about $\wp$? Use your observation to show that e_1 is real.

(d) By (b), it follows that e_1 and e_2 are real. Which one is positive?

(e) Conclude that the elliptic curve $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ corresponding to L is of the form

$$y^2 = 4x(x-a)(x+a)$$

for some real number a.