Lecture Notes from Math 311, Spring 2019

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Contents

Complex Analysis: The study of $\underbrace{\text{holomorphic}}_{\text{complex differentiable on } \Omega}$ / $\underbrace{\text{meromorphic}}$ functions $f : \Omega \to \mathbb{C}$

open subset of $\mathbb{C}$

complex numbers

holomorphic away from a discrete set of isolated points in Ω

f is holomorphic at $z \in \Omega$ if $\underbrace{\lim_{h \to 0}} \dfrac{f(z+h) - f(z)}{h}$ exists

- complex limit
- $h \to 0$ in all possible directions!

Applications:

- analytic continuation (Riemann hypothesis, prime number theorem)
- algebraic geometry (elliptic curves)
- conformal mappings (conformal transforms of harmonic functions are harmonic — fields determined by potentials)

...

To understand complex functions, we'd better start with the

Complex Numbers

$\mathbb{C}$ starts life as the 2-dimensional real vector space spanned by $1, i$, so an element of $\mathbb{C}$ is of the form $a \cdot 1 + b \cdot i =: a + bi$ where $a, b \in \mathbb{R}$, and

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

for $c, d \in \mathbb{R}$.

Multiplication of basis elements:

$\cdot$	1	i
1	1	i
i	i	$? = i^2$

We know the punchline:

$$\boxed{i^2 = -1}.$$

Thus $$(a+bi)\cdot(c+di) = ac + adi + bci + bdi^2$$
$$= (ac-bd) + (ad+bc)i.$$

<u>Note</u> $\cancel{bi = (b+0i)\cdot(0+i) =}$

$$ib = (0+i)(b+0i) = 0 + bi = bi$$

so $a + bi = a + ib$.

<u>Thm</u> $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$ is a field:

- commutative, associative addition
- commutative, associative multiplication
- additive, multiplicative units
- additive inverses
- multiplicative inverses in $\mathbb{C} - \{0\}$.
- multiplication distributes over addition.

<u>Pf</u> Review your 112 notes! Trickiest bit: for $z = a+bi$,

$$z^{-1} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = \frac{\bar{z}}{z\bar{z}} \qquad \square$$

<u>Defn</u> If $z = a+bi$, then $\bar{z} = a - bi$ is its <u>complex conjugate</u>.
The <u>modulus</u> (or norm) of z is $|z| = \sqrt{a^2+b^2}$.
The <u>real part</u> of z is $\mathrm{Re}(z) = a = \frac{1}{2}(z+\bar{z})$.
The <u>imaginary part</u> of z is $\mathrm{Im}(z) = b = \frac{1}{2i}(z-\bar{z})$.

<u>Facts</u> If $z, w \in \mathbb{C}$, then

(a) $\bar{\bar{z}} = z$

(b) $z\bar{z} = |z|^2$

(c) $\overline{z+w} = \bar{z} + \bar{w}$

(d) $\overline{zw} = \bar{z}\,\bar{w}$

(e) $|zw| = |z||w|$ and $|\bar{z}| = |z|$

(f) $|z| \geq 0$ with equality iff $z = 0$

(g) $|\mathrm{Re}(z\bar{w})| \le |z||w|$ [Cauchy-Schwarz]

(h) $|z+w| \le |z| + |w|$ [triangle]

Pf (g) Note $\overline{\bar{z}w} = z\bar{w}$ so $z\bar{w} + \bar{z}w = 2\mathrm{Re}(z\bar{w})$.

For $t \in \mathbb{R}$, $0 \le |zt+w|^2 = (zt+w)(\bar{z}t+\bar{w})$

$$= |z|^2 t^2 + 2\mathrm{Re}(z\bar{w})\,t + |w|^2$$

so RHS has at most one root in $\mathbb{R}$

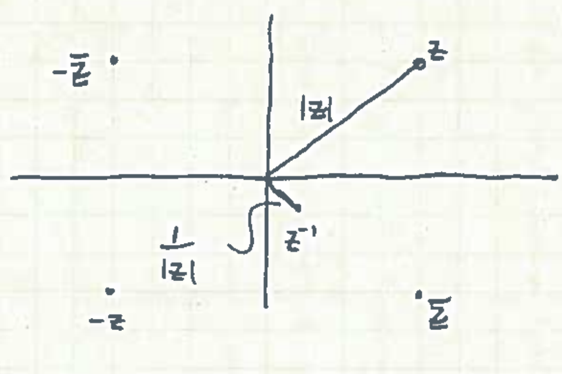
$$\Rightarrow \underbrace{4(\mathrm{Re}(z\bar{w}))^2 - 4|z|^2|w|^2}_{\text{part under } \sqrt{\ } \text{ in quadratic fmla}} \le 0 \qquad \square$$

Pf (h) $|z+w|^2 = (z+w)(\bar{z}+\bar{w})$

$$= |z|^2 + 2\mathrm{Re}(z\bar{w}) + |w|^2$$

$$\le |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2 \qquad \square$$

TPS
- Draw $z, \bar{z}, -z, -\bar{z}, z^{-1}$
- Compute $\dfrac{1}{2+3i}$



Thm If $z = a+bi$, then $\max\{|a|,|b|\} \le |z| \le |a|+|b|$.

Convergence in $\mathbb{C}$

Defn A sequence $\{z_n\}$ of complex numbers converges to $w \in \mathbb{C}$ if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |z_n - w| < \varepsilon$.

In this case, write $\lim z_n = w$ or $z_n \longrightarrow w$.

Note (a) $\lim z_n = w$ iff $\lim |z_n - w| = 0$

(b) For $\{a_n\}, \{b_n\}$ sequences of real numbers with $0 \leq a_n \leq b_n$, if $\lim b_n = 0$, then $\lim a_n = 0$. [squeeze principle]

$N = 6$ for this ε.

Thm A sequence $\{z_n\}$ of complex numbers converges to $w \in \mathbb{C}$ iff $\{\mathrm{Re}(z_n)\}$ converges to $\mathrm{Re}(w)$ and $\{\mathrm{Im}(z_n)\}$ converges to $\mathrm{Im}(w)$.

Pf $|\mathrm{Re}(z_n) - \mathrm{Re}(w)|, |\mathrm{Im}(z_n) - \mathrm{Im}(w)| \underset{(A)}{\leq} |z_n - w| \underset{(B)}{\leq} |\mathrm{Re}(z_n) - \mathrm{Re}(w)| + |\mathrm{Im}(z_n) - \mathrm{Im}(w)|$

Suppose $\lim z_n = w$. Then $\lim |z_n - w| = 0$, so $\mathrm{Re}(z_n) \to \mathrm{Re}(w)$ & $\mathrm{Im}(z_n) \to \mathrm{Im}(w)$ by (A).

Suppose $\lim \mathrm{Re}(z_n) = \mathrm{Re}(w)$ & $\lim \mathrm{Im}(z_n) = \mathrm{Im}(w)$.

Then $\lim z_n = w$ by (B). $\square$

e.g. $\lim \left(\frac{1}{n} + \frac{n^2}{2n^2+1} i \right) = \frac{1}{2} i$.

Thm If $\lim z_n = z$ and $\lim w_n = w$, then $\lim (z_n + w_n) = z + w$ and $\lim (z_n w_n) = zw$. [limits respect addition and mult'n.]

TPS Converse?

Pf Reading (Example 1.2.6). $\square$

Series of complex numbers

A series of complex numbers is a formal sum

$$\sum_{k=0}^{\infty} z_k = z_0 + z_1 + z_2 + \cdots$$

with $z_k \in \mathbb{C}$. Its n-th partial sum is $s_n = \sum_{k=0}^{n} z_k$.

$\sum_{k=0}^{\infty} z_k$ converges to w if $s_n \to w$.

Thm [Term test] If $\sum_{k=0}^{\infty} z_k$ converges, then $z_k \to 0$. $\square$

e.g. $\sum_{k=0}^{\infty} z^k$ converges to $\frac{1}{1-z}$ for $|z|<1$; diverges if $|z| \geq 1$.

Indeed, $s_n = \frac{1-z^{n+1}}{1-z}$ so $\left|s_n - \frac{1}{1-z}\right| = \frac{|z|^{n+1}}{|1-z|} \to 0$.

Defn $\sum_{k=0}^{\infty} z_k$ converges absolutely if $\sum_{k=0}^{\infty} |z_k|$ converges.

Thm If $\sum_{k=0}^{\infty} z_k$ converges absolutely, then it converges, and

$$\left|\sum_{k=0}^{\infty} z_k\right| \leq \sum_{k=0}^{\infty} |z_k|$$

Pf Reading [check on real, im parts; use Δ id]. $\square$

e.g. $\sum_{k=1}^{\infty} \frac{1}{k+k^2 i}$ converges:

$|k+k^2 i| \geq k^2$ so $\frac{1}{|k+k^2 i|} \leq \frac{1}{k^2}$. The p-series $\sum_{k=1}^{\infty} k^{-2}$ converges, so so does $\sum_{k=1}^{\infty} \frac{1}{|k+k^2 i|}$ by the comparison test. Thus $\sum_{k=1}^{\infty} \frac{1}{k+k^2 i}$ is absolutely convergent.

Power Series

A complex power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $a_n, z_0 \in \mathbb{C}$, z a variable. We say this series is centered at z_0; it defines a function $\Omega \to \mathbb{C}$ for

$\Omega = \{z \in \mathbb{C} \mid \text{series converges at } z\}$.

Let $D_r(z_0) = \{z \in \mathbb{C} \mid |z-z_0| < r\}$

$\overline{D}_r(z_0) = \{z \in \mathbb{C} \mid |z-z_0| \le r\}$.

From Math 112, know $\exists R \in \mathbb{R}_{>0} \cup \{\infty\}$ s.t.

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for $z \in D_R(z_0)$

diverges for $z \notin \overline{D}_R(z_0)$;

call R the radius of convergence of the power series.

TPS How do we find R?

A The ratio test on the absolute series!

e.g. What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!\,z^n}{n!\,(2n)!}$?

Let $a_n = \dfrac{(3n)!\,z^n}{n!\,(2n)!}$. Then $\dfrac{|a_{n+1}|}{|a_n|} = \dfrac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+2)(n+3)}|z|$

$$= \frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}|z| \longrightarrow \frac{27}{4}|z|$$

Thus $\sum a_n$ is absolutely convergent for $|z| < \frac{4}{27}$, divergent for $|z| > \frac{4}{27}$, and $R = \frac{4}{27}$.

The Exponential Function

Defn $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ is the complex exponential function.

$$z \longmapsto e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note Converges on $\mathbb{C}$ by ratio test.

Thm $e^{z+w} = e^z e^w$

Pf The binomial theorem tells us $(z+w)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j}$.

Thus
$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} z^j w^{n-j}$$

$$= \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{z^j w^k}{j!k!}$$

$$= e^z e^w \quad \text{(expanding product of power series)}. \qquad \square$$

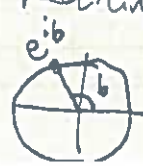
The last step is valid b/c absolutely convergent power series have product $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \sum_{j+k=n} a_j b_k$.

Now $e^{a+bi} = e^a e^{ib}$. We understand e^a from Math 111.
For e^{ib}, note $e^{ib} = \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k+1}}{(2k+1)!}$

TPS Fmla for i^n.

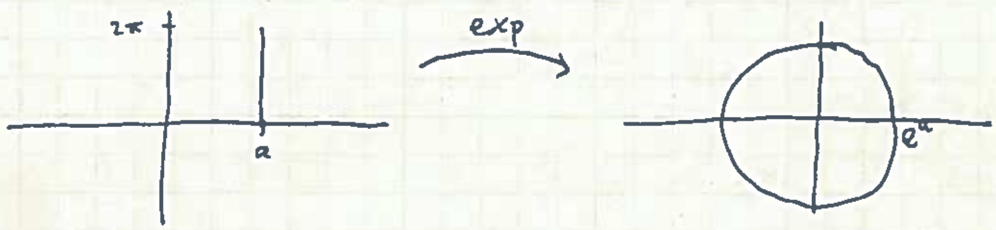
$$\boxed{e^{ib} = \cos b + i \sin b}$$

Note e^{ib} is on the unit circle, b radians ccw from positive real axis.



Visualizing exp:

- exp is $2\pi i$ periodic: $e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1$
- So if we understand exp on $\{z \in \mathbb{C} \mid \cancel{0 \leq}\ -\pi < \operatorname{Im}(z) \leq \cancel{2\pi}\ \pi\}$, then we understand exp on all of $\mathbb{C}$ (rolling up the plane).
- The segment $a + \cancel{[0,2\pi)i}\ (-\pi,\pi]i$ becomes $e^a e^{\cancel{[0,2\pi)i}\ (-\pi,\pi]i}$ = circle of radius e^a centered at 0.



Properties of exp:

__Thm__ (a) $e^z \neq 0 \quad \forall z \in \mathbb{C}$

(b) $|e^z| = e^{\operatorname{Re}(z)}$

(c) $|e^z| \leq e^{|z|}$

(d) $e^z = 1$ iff $z = 2\pi n i$ for some integer n.

__TPS__ · Derive the angle addition fmlae for cos, sin using $e^{i\theta} = \cos\theta + i\sin\theta$

· Create fmlae for $\cos(n\theta)$, $\sin(n\theta)$ for $n \in \mathbb{N}$.

__Polar Form__ $\forall\, 0 \neq z \in \mathbb{C}\ \exists!\, r > 0,\ \theta \in \cancel{[0,2\pi)}\ (-\pi,\pi]$ s.t. $\boxed{z = re^{i\theta}}$.

For $z = 0$, take $r = 0$ and any θ.

__Defn__ The __argument__ of $z = re^{i\theta}$ is $\theta + 2\pi\mathbb{Z} = \{\theta + 2\pi n \mid n \in \mathbb{Z}\}$, denoted $\arg(z)$.

__TPS__ Multiply + divide cpx #s in polar form.

Thm If $z = re^{i\theta} \neq 0$, then z has exactly n n-th roots,
$$r^{1/n} e^{i(\theta/n + 2\pi k/n)}, \quad k = 0, 1, \dots, n-1.$$

e.g. The "n-th roots of $\underbrace{\text{unity}}_{\text{i.e. } 1}$" are $e^{2\pi k i/n}$, $k = 0, \dots, n-1$.

5th roots of 1.

Complex Logarithm

$$z = re^{i\theta} = e^{\log r + i\theta}$$ so we would like to define

($\log$: real natural log)

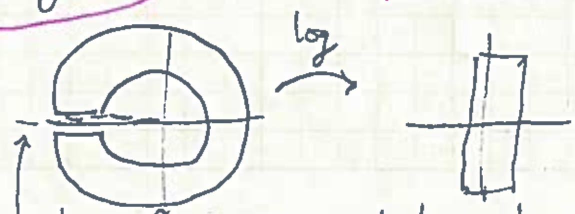
$$\log z = \log r + i\theta = \log|z| + i\underbrace{\arg(z)}_{\text{infinitely many values!}}$$

Solution Restrict $\theta = \arg(z)$ to lie in a half-open interval $I \subseteq \mathbb{R}$ of length 2π.

Defn Given a half-open interval I of length 2π, let $\arg_I z$, for $0 \neq z \in \mathbb{C}$, be the element of $\arg(z) \cap I$. Then the function $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $\log(z) = \log|z| + i \arg_I z$ is the branch of the log function defined by I. When $I = (-\pi, \pi]$, call this the principal branch of log.

TPS Let log be some branch of log.

- What is $\log(zw)$? $= \log(z) + \log(w) + 2\pi k i$ for some $k \in \mathbb{Z}$
- $\log(1)$?
- Is log continuous?



cut line for principal branch

Read <u>Thm 1.4.8</u> for standard properties of log.

<u>Other functions</u> Given a branch of log, define

$$z^{1/n} = e^{(1/n)\log(z)} \text{ for } z \neq 0, \quad 0^{1/n} = 0.$$

In particular, $\sqrt{z} = e^{\frac{1}{2}\log(z)}$

Discontinuities along negative real axis!

Continuity $f : \Omega \to \mathbb{C}$ is *continuous* if either of the following equivalent conditions holds:

① For all $a \in \Omega$, $\lim_{z \to a} f(z) = f(a)$

② For all open $W \subseteq \mathbb{C}$, $f^{-1}(W) \subseteq \Omega$ is open.

Reading: 2.1 (review since equivalent to continuity of function $\Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$).

The Complex Derivative

Defn Let f be a fn defined on a nbhd of $z \in \mathbb{C}$. If $\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ exists, we denote it $f'(z)$ and say f is *complex diff'l* at z with *complex derivative* $f'(z)$. If f is defined and diff'l at every point of an open set U, then call f *analytic* on U.

e.g. · For $f(z) = z$, $\lim_{h \to 0} \frac{z+h-z}{h} = \lim_{h \to 0} 1 = 1$, so $f'(0) = 1$.

· For $g(z) = \bar{z}$, $\lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h \to 0} e^{-2i\theta}$

for $h = re^{i\theta}$. This limit does not exist! (Different values along any ray emanating from 0.) So g is not complex differentiable at $z = 0$, despite $G : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, -y)$ being infinitely diff'l!

Prop exp is analytic on $\mathbb{C}$ and $\exp' = \exp$.

Pf Have $\dfrac{e^{z+h}-e^z}{h} = e^z\dfrac{e^h-1}{h}$, so suffices to show

$\dfrac{e^h-1}{h} \to 1$ as $h \to 0$. Expanding its power series gives

$$\frac{e^h-1}{h}-1 = \frac{e^h-1-h}{h} = \frac{\frac{h^2}{2}+\frac{h^3}{3!}+\cdots}{h} = \frac{h}{2}+\frac{h^2}{3!}+\cdots \to 0,$$

so $(e^z)' = e^z$. $\square$

<u>Basic properties</u>

- If $f'(a)$ exists, then f is cts at a.
- $(f+g)'(z) = f'(z)+g'(z)$ when RHS makes sense.
- $(fg)'(z) = f'(z)g(z)+f(z)g'(z)$ —— " ——
- If $g(z)\neq 0$ and $g'(z)$ exists, then $(1/g)'(z) = -g'(z)/g^2(z)$.
- $(f/g)'(z) = \dfrac{f'(z)g(z)-g'(z)f(z)}{g^2(z)}$
- $(f\circ g)'(a) = f'(g(a))\,g'(a)$

Proofs exactly mirror those from Math 112.

<u>Cauchy-Riemann Equations</u>

For $f:\Omega\to\mathbb{C}$ write $f(x+iy) = u(x,y)+iv(x,y)$
to consider f a function on a subset of $\mathbb{R}^2$ with components u,v.
~~Analyticity~~ Complex diff'lity at z_0 is equivalent to the existence of $c=a+ib$ s.t.

① $$\lim_{z\to z_0}\frac{1}{z-z_0}\left(f(z)-f(z_0)-c(z-z_0)\right) = 0.$$

The function $F: (x,y) \longmapsto (u(x,y), v(x,y))$ is diff'l at (x_0,y_0) iff $\exists\, M = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ s.t.

$$\text{②} \quad \lim_{(x,y)\to(x_0,y_0)} \frac{1}{|(x-x_0, y-y_0)|} \left| F(x,y) - F(x_0,y_0) - M\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \right| = 0.$$

It might (but may not) happen that

$$\text{③} \quad M = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \in \mathbb{C}. \quad (\text{so } r = -q,\ s = p)$$

From HW, you know that if $M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then (for $c = a+ib$)

$$cz = (ax-by) + i(bx+ay) \longleftrightarrow \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$$

Thus ① holds iff ② & ③ hold (and in this case, $p = a$, $q = b$).

<u>Thm</u> [Cauchy-Riemann equations]

$f: \Omega \longrightarrow \mathbb{C}$ is ~~analytic~~ cpx diff'l at z_0 iff "$F^{(u,v)}$" is diff'l at (x_0,y_0) with $\boxed{u_x = v_y\,,\ u_y = -v_x}$ at (x_0,y_0).

In this case, $f' = u_x + iv_x = v_y - iu_y$. $\square$

<u>e.g.</u> With $z = x+iy$, $e^z = e^x(\cos y + i\sin y)$ so $u = e^x\cos y$, $v = e^x \sin y$. Further,

$$u_x = e^x\cos y = v_y$$
$$u_y = -e^x\sin y = -v_x$$

as expected since e^z is analytic.

Harmonic Functions

In a bit, we'll prove that analytic fns have cts cpx derivatives of all orders. Assuming this result for a moment, we have

Thm If $f:\Omega \to \mathbb{C}$ has $f = u + iv$ and is analytic on Ω, then $\underbrace{u_{xx} + u_{yy} = 0}_{\text{Say } u \text{ is harmonic}}$. and $v_{xx} + v_{yy} = 0$.

Pf $u_{xx} = (u_x)_x \underset{CR}{=} (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}$

$v_{xx} = (v_x)_x \underset{CR}{=} (-u_y)_x = -(u_x)_y = -(v_y)_y = -v_{yy}$.

Defn If u, v are harmonic functions s.t. $f = u + iv$ is analytic, then we call u, v <u>harmonic conjugates</u> of one another.

Facts
- If it exists, the harmonic conjugate of u is unique up to an additive constant
- u is conjugate to v iff v is conjugate to u.

$\boxed{2F}$

Contour integrals

A <u>curve</u> (or <u>contour</u>) in $\mathbb{C}$ is a continuous function $\gamma: I \to \mathbb{C}$ where $I = [a,b] \subseteq \mathbb{R}$ is an interval.

For $c \in I$, define $\gamma'(c) = \lim_{t\to c} \frac{\gamma(t) - \gamma(c)}{t - c}$. If $\gamma(t) = x(t) + iy(t)$, then $\gamma'(c) = x'(c) + iy'(c)$.

Call γ <u>continuously differentiable</u> on I if diff'l at all $c \in I^\circ$ with γ' cts. In this case, write $\gamma \in C^1(I)$.

γ is piecewise smooth if $\exists\, a = a_0 < a_1 < \cdots < a_n = b$ s.t. γ is C^1 on each $[a_{j-1}, a_j]$. A curve that is piecewise smooth will be called a path.

e.g. $\cdot\ \gamma : [0, 2\pi] \longrightarrow \mathbb{C}$
$\quad t \longmapsto re^{it}$

$\cdot\ \gamma : [0,1] \longrightarrow \mathbb{C}$ for some fixed $z, w \in \mathbb{C}$
$\quad t \longmapsto (1-t)z + tw$

TPS What do these parametrize?

Riemann Integral of cpx-valued Fns:

$f : I \longrightarrow \mathbb{C}$ for g, h real valued, $I = [a,b]$
$\quad t \longmapsto g(t) + ih(t)$

define $\displaystyle\int_a^b f = \int_a^b g + i\int_a^b h$.

Thm The Riemann integral of cpx fns is $\mathbb{C}$-linear. $\square$

Thm $\displaystyle\int_a^b f = \int_a^c f + \int_c^b f$. $\square$

Thm $\displaystyle\left|\int_a^b f\right| \le \int_a^b |f|$

Pf (by trick!) Set $w = \int_a^b f \in \mathbb{C}$. If $w = 0$, done. If $w \ne 0$, set $u = \dfrac{\overline{w}}{|w|}$, so $uw = |w|$. Thus

$|w| = \left|\int_a^b f\right| = u\int_a^b f = \int_a^b uf \in \mathbb{R}$. Thus $\int_a^b \operatorname{Im}(uf) = 0$

and $\left|\int_a^b f\right| = \int_a^b \operatorname{Re}(uf) \le \int_a^b |uf| = \int_a^b |f|$ since $|u| = 1$. $\square$

Integration along a path:

If $f : \Omega \to \mathbb{C}$ continuous and $\gamma(I) \subseteq \Omega$, then $(f \circ \gamma) \cdot \gamma'$ is defined on I (except at finitely many discontinuities of γ'), and is piecewise cts, hence Riemann integrable.

<u>Defn</u> For $\gamma : [a,b] \to \mathbb{C}$ a path, $f : \Omega \to \mathbb{C}$ cts with $\gamma([a,b]) \subseteq \Omega$, the <u>integral</u> of f over γ is

$$\int_\gamma f := \int_a^b (f \circ \gamma) \cdot \gamma' .$$

$$\left(\text{or } \int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt\right)$$

Think: $z \rightsquigarrow \gamma(t)$

$dz \rightsquigarrow \gamma'(t)\, dt$.

<u>e.g.</u> $f(z) = z$

$\gamma : [0, 2\pi] \to \mathbb{C}$ $\qquad \gamma'(t) = ire^{it}$

$t \longmapsto re^{it}$

$$\int_\gamma z\, dz = \int_0^{2\pi} re^{it} \cdot ire^{it}\, dt$$

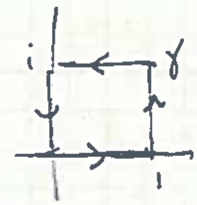
$$= ir^2 \int_0^{2\pi} e^{2it}\, dt$$

$$= ir^2 \int_0^{2\pi} (\cos(2t) + i\sin(2t))\, dt$$

$$= 0 .$$

e.g. $\gamma : [0,4] \longrightarrow \mathbb{C}$

$$t \longmapsto \begin{cases} t & \text{if } t \in [0,1) \\ 1 + i(t-1) & \text{if } t \in [1,2] \\ 1 - (t-2) + i & \text{if } t \in [2,3) \\ (1-(t-3))i & \text{if } t \in [3,4) \end{cases}$$



$\int_\gamma x \, dz$:

$$\int_0^1 \mathrm{Re}(\gamma)\gamma' = \int_0^1 t \, dt = \tfrac{1}{2}$$

$$\int_1^2 \mathrm{Re}(\gamma)\gamma' = \int_1^2 i \, dt = i$$

$$\int_2^3 \mathrm{Re}(\gamma)\gamma' = \int_2^3 (3-t)(-1) \, dt = -\tfrac{1}{2}$$

$$\int_3^4 \mathrm{Re}(\gamma)\gamma' = \int_3^4 0 \, dt = 0$$

$$\Longrightarrow \int_\gamma x \, dz = i .$$

Properties of Contour Integrals

Thm [Independence of Parametrization]

Let $\gamma_1 : [a,b] \to \mathbb{C}$ be a path and $\alpha : [c,d] \to [a,b]$ a smooth function with $\alpha(c) = a$, $\alpha(d) = b$. If $\gamma_2 = \gamma_1 \circ \alpha$, then

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

for all f defined and cts on a set containing $\gamma_1([a,b]) = \gamma_2([c,d])$.

Pf By chain rule, $\gamma_2' = (\gamma_1' \circ \alpha) \cdot \alpha'$. Thus

$$\int_{\gamma_2} f = \int_c^d (f \circ \gamma_2) \cdot \gamma_2'$$

$$= \int_c^d (f \circ \gamma_1 \circ \alpha) \cdot (\gamma_1' \circ \alpha) \cdot \alpha'$$

$$= \int_a^b (f \circ \gamma_1) \cdot \gamma_1' \qquad [\text{change of variables, } u = \alpha]$$

$$= \int_{\gamma_1} f. \qquad \square$$

Ind of param'n, but not of image.
e.g. $\gamma_1(t) = re^{it}$, $\gamma_2(t) = re^{-it}$ both on $[0, 2\pi]$.
Then $\displaystyle\int_{\gamma_1} \frac{dz}{z} = 2\pi i$, $\displaystyle\int_{\gamma_2} \frac{dz}{z} = -2\pi i$.

Need $\alpha(c) = a$, $\alpha(d) = b$!

TPS What are the reasonable interpretations of

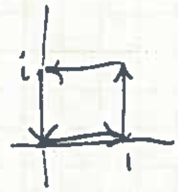
$$\int_{|z|=1} f \ , \quad \int_{\partial\Delta} f \ , \quad \int_{w_1}^{w_2} f$$

for $\Delta \subseteq \mathbb{C}$ triangle, $w_1, w_2 \in \mathbb{C}$? (Preferred or'n is counterclockwise

<u>Defn</u> A <u>closed</u> curve (or path) γ is one such that $\gamma(a)=\gamma(b)$ (for $\gamma:[a,b]\to\mathbb{C}$).

<u>Preview of Cauchy's Thm</u> If $\gamma: I\to U\subseteq\mathbb{C}$ is a closed path, f is analytic on U, and γ does not "go around any holes in U," then $\int_\gamma f=0$.

<u>e.g.</u> $f(z)=z$, γ:



$\int_\gamma f=0$.

<u>Operations on Contour Integrals</u>

$\gamma_1+\gamma_2$

Join of two paths when γ_1 ends where γ_2 starts

Simple enough, but what if the parameter intervals don't match up?

<u>A</u> $\alpha:[c,d]\longrightarrow[a,b]$ is smooth with $\alpha(c)=a$, $\alpha(d)=b$

$$t\longmapsto a+\frac{b-a}{d-c}(t-c)$$

Thus we can always reparametrize to get whatever parameter interval we want.

Now if $\gamma_1:[a,b]\to\mathbb{C}$, $\gamma_2:[b,c]\to\mathbb{C}$ with $\gamma_2(b)=\gamma_2(b)$, define

$$\gamma_1+\gamma_2(t)=\begin{cases}\gamma_1(t) & \text{if } t\in[a,b]\\ \gamma_2(t) & \text{if } t\in[b,c].\end{cases}$$

Define the *reverse* of $\gamma : [a,b] \to \mathbb{C}$ by

$$-\gamma : [a,b] \longrightarrow \mathbb{C} \qquad\qquad -\gamma(t) \neq -(\gamma(t))$$
$$t \longmapsto \gamma(a+b-t)$$

For closed simple paths, call counterclockwise orientation to be *positive*; clockwise to be *negative*.

Thm $\gamma, \gamma_1, \gamma_2$ paths, γ_1 ending where γ_2 starts, f, g cts fns $\Omega \to \mathbb{C}$ with $\operatorname{im}\gamma, \operatorname{im}\gamma_1, \operatorname{im}\gamma_2 \subseteq \Omega$, $\lambda \in \mathbb{C}$. Then

(a) $\int_\gamma \lambda f + g = \lambda \int_\gamma f + \int_\gamma g$

(b) $\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

(c) $\int_{-\gamma} f = -\int_\gamma f$

Length

Defn The *length* of $\gamma : [a,b] \to \mathbb{C}$ is $\ell(\gamma) = \int_a^b |\gamma'(t)|\, dt$

Thm $\gamma : I \to \mathbb{C}$ path, $f : \Omega \to \mathbb{C}$ cts with $\gamma(I) \subseteq \Omega$, $|f(z)| \leq M$ $\forall z \in \gamma(I)$, then

$$\left| \int_\gamma f \right| \leq M \ell(\gamma).$$

Pf $\left| \int_\gamma f \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\, dt \right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, dt$

$$\leq \int_a^b M |\gamma'(t)|\, dt = M \ell(\gamma). \qquad \square$$

Cauchy's Integral Thm for a Triangle

Idea $U \xrightarrow[\text{analytic}]{f} \mathbb{C} \implies \int_\gamma f = 0$

for U "simply connected."

First case: $U = \Delta$

Note: $\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$ if $F' = f$ (HW)

so $\int_\gamma f = 0$ if γ closed and f has a primitive F.

Idea: cpx diff'l fns have linear approx'ns, and linear functions have primitives, so analytic fns are approximated by fns w/ integral 0 around closed paths.

Lemma Let f be cts on a nbhd of $w \in \mathbb{C}$, f cpx diff'l at w. Then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\left| \int_{\partial\Delta} f \right| < \varepsilon d^2$

if Δ is any triangle containing w of diameter $d \le \delta$.

Note: diam(Δ) = longest side length.

Pf Since f is cts on a nbhd of w, $\exists r > 0$ s.t. f is cts on $D_r(w)$.

Know that $\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = f'(w)$ exists, so $\exists\, 0 < \delta < r$ s.t.

$$|z - w| < \delta \implies \left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \frac{\varepsilon}{3}$$

$$\implies | f(z) - f(w) - f'(w)(z - w) | < \frac{\varepsilon}{3} |z - w| \quad \text{for } z \in D_\delta(w).$$

For Δ a triangle of diam $d \le \delta$, containing w, set

$$I = \int_{\partial\Delta} f$$

Then

$$I = \underbrace{\int_{\partial\Delta} \big(f(w) + f'(w)(z-w)\big)\,dz}_{\substack{=0 \text{ b/c linear} \\ \text{in } z \text{ so has a} \\ \text{primitive}}} + \int_{\partial\Delta} \big(f(z) - f(w) - f'(w)(z-w)\big)\,dz$$

so $$I = \int_{\partial\Delta} \big(f(z) - f(w) - f'(w)(z-w)\big)\,dz$$

$$\Rightarrow |I| \le \int_{\partial\Delta} |f(z) - f(w) - f'(w)|\,dz$$

$$< \int_{\partial\Delta} \frac{\varepsilon}{3}|z-w|\,dz$$

$$< \int_{\partial\Delta} \frac{\varepsilon d}{3}\,dz$$

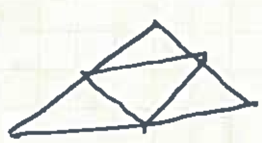
$$\le \frac{\varepsilon d}{3}(3d) = \varepsilon d^2. \qquad \square$$

<u>Thm</u> Let f be a function which is analytic in an open set U, and suppose Δ is a triangle contained in U. Then

$$\int_{\partial\Delta} f = 0.$$

<u>Pf</u> Set $I = \int_{\partial\Delta} f$. We show $I = 0$ by showing $|I| < \varepsilon$ $\forall \varepsilon > 0$.

Let $\varepsilon > 0$. Subdivide Δ into four triangles via side midpts:



Get all subtriangles similar to Δ.

Have

Let Δ_1 be a subtriangle s.t. $|I_1| \geq |I|/4$ for $I_1 = \int_{\partial\Delta_1} f$

Note that if $\text{diam}(\Delta) = h$ then $\text{diam}(\Delta_1) = \frac{h}{2}$.

Repeat the subdivision with Δ_1 to get Δ_2 of diameter $\frac{h}{2^2}$ and with $|I_2| \geq |I|/4^2$, $I_2 = \int_{\partial\Delta_2} f$.

Proceeding by induction, get Δ_n of diameter $\frac{h}{2^n}$

(*) with $|I_n| \geq |I|/4^n$, $I_n = \int_{\partial\Delta_n} f$.

Then $\Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots$ is a nested sequence of compact subsets of $\mathbb{C} \Rightarrow \exists\, w \in \bigcap_{n\geq 1} \Delta_n$.

By the lemma (with $\frac{\varepsilon}{h^2}$ in place of ε), we conclude that $\exists\, \delta > 0$ s.t. the integral of f around any triangle containing w of diameter $d \leq \delta$, is less than $d^2\varepsilon/h^2$.

Now take $n \gg 0$ so that $h_n = \frac{h}{2^n} < \delta$. Then

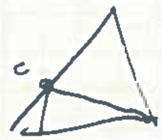
$$|I_n| < \frac{h_n^2}{h^2}\varepsilon = \frac{\varepsilon}{4^n}$$

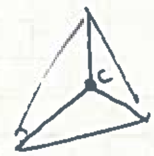
Combined with (*), $|I| \leq 4^n|I_n| < \varepsilon$. Hence $I = 0$. $\square$

<u>Thm</u> The same, but f cts on U, analytic on $U \setminus \{c\}$ for some exceptional pt $c \in \Delta$.

<u>Pf</u> Let $\varepsilon > 0$. If c is a vx, subdivide Δ into smaller & smaller triangles in such a way that the one containing c has circumference $< \frac{\varepsilon}{M}$, M = max'l value of $|f|$ on Δ. $|\int_{\Delta'} f| < \varepsilon$ for $c \in \Delta'$.

$\int_{\text{other } \Delta\text{'s}} f = 0$ by previous thm so $|\int_\Delta f| < \varepsilon \ \forall \varepsilon > 0$. ✓

Other cases: 



$\square$

Cauchy's Thm for a convex set

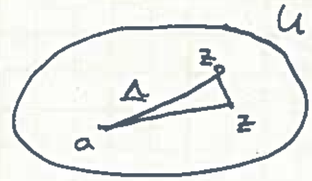
Defn $C \subseteq \mathbb{C}$ is convex if $\forall a, b \in C$, the line segment joining a & b is contained in C:

Thm Let U be a convex open set and suppose f is a cts function on U and has the property that $\int_{\partial\Delta} f = 0$ for all triangles $\Delta \subseteq U$. If $a \in U$ is fixed and

$$F: U \to \mathbb{C}, \quad z \longmapsto \int_a^z f(w)\,dw$$

then $F' = f$ on U.

Pf For $z, z_0 \in U$ consider



By convexity, $\Delta \subseteq U$.

Take $\partial\Delta$ to be the path a to z to z_0 to a. Then

$$0 = \int_{\partial\Delta} f = \int_a^z f + \int_z^{z_0} f + \int_{z_0}^a f = F(z) - F(z_0) - \int_{z_0}^{z} f.$$

Thus
$$F(z) - F(z_0) = \int_{z_0}^z f = \int_{z_0}^z f(z_0)\,dw + \int_{z_0}^z \big(f(w) - f(z_0)\big)\,dw$$
$$= f(z_0)(z - z_0) + \int_{z_0}^z \big(f(w) - f(z_0)\big)\,dw$$

$$\Longrightarrow \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{z_0}^z \big(f(w) - f(z_0)\big)\,dw$$

To show $F'(z_0) = f(z_0)$, need to show RHS $\longrightarrow 0$ as $z \to z_0$.
Let $\varepsilon > 0$. By continuity, $\exists \delta > 0$ s.t. $|f(w) - f(z_0)| < \varepsilon$ when $|w - z_0| < \delta$.
If $|z - z_0| < \delta$, then $|w - z_0| < \delta$ $\forall w \in [z, z_0]$ so $|f(w) - f(z_0)| < \varepsilon$ $\forall w \in [z, z_0]$.

Then $\left| \int_{z_0}^{z} (f(w)-f(z_0))\,dw \right| < \varepsilon |z-z_0|$

$$\Rightarrow \left| \frac{1}{z-z_0} \int_{z_0}^{z} (f(w)-f(z_0))\,dw \right| < \varepsilon \quad \text{for } |z-z_0|<\delta$$

Thus $\displaystyle\lim_{z\to z_0} \frac{1}{z-z_0}\int_{z_0}^{z}(f(w)-f(z_0))\,dw = 0$, as desired. $\square$

<u>Thm</u> Let U be a convex open set and suppose f is analytic on U, except possibly at one pt, and cts on U. Then $\int_\gamma f = 0$ for all closed paths γ in U.

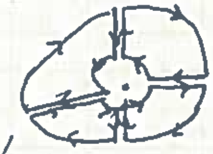
<u>Pf</u> f has an primitive fn on U. $\square$

<u>Cor</u> (TPS) If U is convex open, f analytic on U, $a,b\in U$, then $\int_\gamma f$ is the same for all γ starting at a, ending at b.

$\int_{\gamma_1-\gamma_2} f = 0 \ldots$

<u>Prop</u> / <u>e.g.</u> $\displaystyle\int_\gamma \frac{dz}{z} = 2\pi i$ for

<u>Pf</u>



$$\Rightarrow \int_\gamma \frac{dz}{z} = \int_{\odot} \frac{dz}{z} = 2\pi i.$$

each piece inside a convex open set on which $\frac{1}{z}$ is analytic $\Rightarrow \oint \frac{dz}{z} = 0$ around these. $\square$

<u>Index</u> (of a path around a point)

<u>Defn</u> $\gamma: I \to \mathbb{C}$ any closed path in $\mathbb{C}$, $z\in\mathbb{C}\setminus\gamma(I)$. The <u>index</u> of z wrt γ is $\displaystyle \mathrm{Ind}_\gamma(z) := \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}$.

Thm If γ is a closed path in $\mathbb{C}$ with parameter interval $I=[a,b]$ then $\mathrm{Ind}_\gamma(z)$ is an integer-valued fn of $z \in \mathbb{C}\setminus\gamma(I)$.

Pf Take $z_0 \in \mathbb{C}\setminus\gamma(I)$. Have $\gamma(a)=\gamma(b)$. Define $\lambda : I \to \mathbb{C}$ by $\lambda(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s)-z_0}\,ds$. Then $\lambda(a)=0$ and $\lambda(b) = 2\pi i\,\mathrm{Ind}_\gamma(z)$.

Suffices to show $e^{\lambda(b)}=1$ (b/c this gives $\lambda(b)=2\pi i n$, $n\in\mathbb{Z}$).

By FTC, $\lambda'(t) = \frac{\gamma'(t)}{\gamma(t)-z_0}$ while

$$\left(e^{\lambda(t)}\right)' = e^{\lambda(t)}\lambda'(t) = e^{\lambda(t)}\frac{\gamma'(t)}{\gamma(t)-z_0}$$

Thus $$\left(\frac{e^{\lambda(t)}}{\gamma(t)-z_0}\right)' = \frac{1}{(\gamma(t)-z_0)^2}\left(e^{\lambda(t)}\frac{\gamma'(t)}{\gamma(t)-z_0}(\gamma(t)-z_0) - e^{\lambda(t)}\gamma'(t)\right) = 0$$

Hence $\frac{e^{\lambda(t)}}{\gamma(t)-z_0}$ is constant. In particular, $\frac{e^{\lambda(t)}}{\gamma(t)-z_0} = \frac{e^{\lambda(a)}}{\gamma(a)-z_0}$ $= \frac{1}{\gamma(a)-z_0}$ $\forall t\in[a,b]$. Setting $t=b$, get

$$e^{\lambda(b)} = \frac{\gamma(b)-z_0}{\gamma(a)-z_0} = 1, \text{ as desired.} \qquad \square$$

Thm $U\subseteq\mathbb{C}$ convex open, $f: U\to\mathbb{C}$ analytic, $\gamma: I\to U$ closed path. Then $\mathrm{Ind}_\gamma(z)\,f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$ $\forall z\in U\setminus\gamma(I)$.

Pf Define $g: U\times U \to \mathbb{C}$ by $g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w\neq z \\ f'(z) & \text{o/w} \end{cases}$

For any $z\in U$, $0 = \int_\gamma g(z,w)\,dw = \int_\gamma \frac{f(w)}{w-z}\,dw - \int_\gamma \frac{f(z)}{w-z}\,dw$

$$= \int_\gamma \frac{f(w)}{w-z}\,dw - 2\pi i\,\mathrm{Ind}_\gamma(z)\,f(z)$$

TPS Why does $g(z,-)$ satisfy Cauchy's Thm hypotheses? $\square$

Interpretation For $\mathrm{Ind}_\gamma(z) \neq 0$ (z "inside" γ),
values of f at z determined by values
of f ~~on~~ path.

Properties of the Index Function

Goal $Ind_\gamma : \mathbb{C} \setminus \gamma(I) \longrightarrow \mathbb{Z}$ is constant on the connected components of $\mathbb{C} \setminus \gamma(I)$

Connected Sets

Defn A set $E \subseteq \mathbb{C}$ is separated if $\exists$ a pair A, B of open subsets of $\mathbb{C}$ s.t. $E \subseteq A \cup B$, $A \cap E, B \cap E \neq \emptyset$, $A \cap B = \emptyset$. Say that A, B separate E. If E is not separated, then call it connected.

A maximal connected subset containing $z \in E$ is called the (connected) component of z. Two conn'd components of E are identical or disjoint (why?) & conn'd components partition E.

Call $E \subseteq \mathbb{C}$ path connected if every two points in E can be joined with a path in E.

Thm Let $U \subseteq \mathbb{C}$ be open. Then
(a) each component of U is open
(b) U is connected iff U is path connected.

Pf (a) Let $V \subseteq U$ be a conn'd component containing z.
Then $V = \bigcup$ conn'd subsets containing z. Since U open, $\exists r > 0$ with $D_r(z) \subseteq U$. Since $D_r(z)$ conn'd, $D_r(z) \subseteq V$, so V open.

(b) Suppose U conn'd. For $z \in U$, let V_z = pts of U conn'd to z by a path in U. Let $w \in U$. $\exists$ open disc $D = D_r(w) \subseteq U$. Either $D \subseteq V_z$ or $D \subseteq U \setminus V_z$. $\Rightarrow V_z, U \setminus V_z$ open $\subseteq U$ with union U. Since U conn'd, one of them is empty. $z \in V_z \neq \emptyset$
$\Rightarrow U \setminus V_z = \emptyset \Rightarrow V_z = U$ so U path conn'd

Suppose U path conn'd, sep'd by A, B. Then

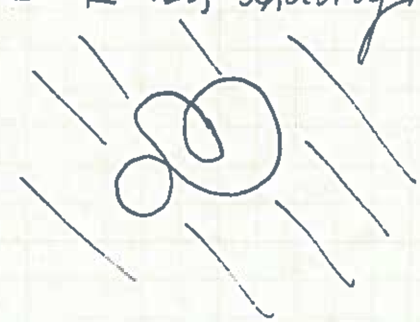
$$f: U \longrightarrow \mathbb{C} \qquad u \longmapsto \begin{cases} 1 & u \in A \\ 0 & u \in B \end{cases}$$

cts. Since U is path conn'd, $\exists$ path γ connecting $a \in A$ to $b \in B$. Then $f \circ \gamma$ cts $\not\models$ IVT. $\square$

<u>Note</u> • For arbitrary spaces, path conn'd $\Rightarrow$ conn'd, but not the converse (topologist's sine curve )

• For K compact, $\mathbb{C} \setminus K$ is the union of its conn'd cpts, each of which is open and path conn'd.

<u>Thm</u> If $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \setminus K$ has exactly one unbounded component.



<u>Pf</u> $K \subseteq \overline{D}_R(0)$ since closed and bdd.

$\Rightarrow$ $\underbrace{\mathbb{C} \setminus K}_{\text{open}} \supseteq \underbrace{\mathbb{C} \setminus \overline{D}_R(0)}_{\text{connected hence contained in a component of } \mathbb{C}\setminus K}$.

$\Rightarrow$ all other components of $\mathbb{C} \setminus K$ contained in $\overline{D}_R(0)$ hence bounded. $\square$

<u>Thm</u> If $\gamma : I \longrightarrow \mathbb{C}$ is a closed path, then $\mathrm{Ind}_\gamma(z)$ is constant on each component of $\mathbb{C} \setminus \gamma(I)$, and is 0 on the unbounded component.

<u>Pf</u> Take $z_0 \in \mathbb{C} \setminus \gamma(I)$ and $D_R(z_0) \subseteq \mathbb{C} \setminus \gamma(I)$. First show that on some smaller disc centered at z_0, Ind_γ is constant. Suppose $0 < r < R$, $z \in D_r(z_0)$. Then

$$\mathrm{Ind}_\gamma(z) - \mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} - \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z_0}$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{z - z_0}{(w - z)(w - z_0)}\, dw .$$

For $w \in \gamma(I)$, $|w-z_0| \geq R$, $|w-z| \geq R-r$.

Since $|z-z_0|<r$, $\quad \left| \dfrac{z-z_0}{(w-z)(w-z_0)} \right| \leq \dfrac{r}{R(R-r)}$

$\Rightarrow \quad |\text{Ind}_\gamma(z) - \text{Ind}_\gamma(z_0)| \leq \dfrac{r\,\ell(\gamma)}{2\pi R(R-r)} < 1$ for r suff small

But & $\in \mathbb{Z}$, so in fact are equal on $D_r(z_0)$.

Let A be a component of $\mathbb{C}-\gamma(I)$, and for each $n \in \mathbb{Z}$ let $V_n = \{z \in A \mid \text{Ind}_\gamma(z) = n\}$. Each $V_n \subseteq A$ open, by above. $\bigcup_{m \neq n} V_m$ open as well with $V_n \cup \bigcup_{m \neq n} V_m = A$. Since A is conn'd, one of the sets is empty. Thus $V_n \neq \emptyset \Rightarrow V_n = A$ and Ind_γ constant on components of $\mathbb{C}-\gamma(I)$.

Remains to show $\text{Ind}_\gamma(z) = 0$ on unbdd cpt of $\mathbb{C}-\gamma(I)$.

Take D open disc $\supseteq \gamma(I)$, $z_0 \in \mathbb{C}-D$, so z_0 in unbdd cpt.

$$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z_0} = 0 \quad \text{since } D \text{ convex} \supseteq \gamma(I)$$

with $\dfrac{1}{w-z_0}$ analytic on D. $\square$

e.g. γ tracing n times around z_0 in circle of radius r:

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto z_0 + re^{int}.$$

Then $$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)-z_0}\,dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} in\,dt = n.$$

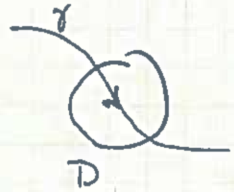
$\text{Ind}_\gamma(V_{z_0}) = \{n\}$, $\quad \text{Ind}_\gamma(\mathbb{C} - \overline{V}_{z_0}) = \{0\}$.

<u>Defn</u> $\gamma : [a,b] \to \mathbb{C}$ path, D open disc. Say γ simply splits D if

(a) $J = \gamma^{-1}(D) = (c,d) \subseteq [a,b]$ or, in case γ closed with $\gamma(a) = \gamma(b) \in D$,

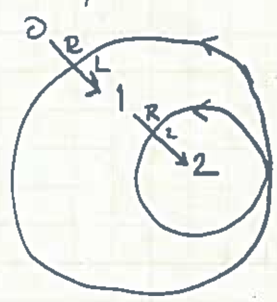
$J = [a,c) \cup (d,b] \subseteq [a,b]$ with $c < d$.

(b) $D \setminus \gamma(J)$ has exactly two components



<u>Thm</u> Let γ be a closed path which simply splits a disc D. Then $\mathrm{Ind}_\gamma(z) = 1 + \mathrm{Ind}_\gamma(w)$ if z is in the left and w in the right component of $D \setminus \gamma(J)$.

<u>e.g.</u>



Defn Let $\{f_n : S \subseteq \mathbb{C} \to \mathbb{C}\}$ be a sequence of functions.
Then (a) $\{f_n\}$ converges pointwise to the function $f : S \to \mathbb{C}$ if $\forall z \in S$, $\{f_n(z)\} \longrightarrow f(z)$.

(b) $\{f_n\}$ converges uniformly on S if $\forall \varepsilon > 0\ \exists N$ s.t.
$$|f_n(z) - f(z)| < \varepsilon \quad \forall n \geq N, z \in S.$$

Note In (a), N can depend on z; in (b) N is independent of z.

Thm If $\{f_n : E \subseteq \mathbb{C} \to \mathbb{C}\} \xrightarrow[\text{unif}]{} f$ and each f_n is continuous, then f is continuous.

Pf Take $z_0 \in E$. Given $\varepsilon > 0$ choose N s.t. $n \geq N \Longrightarrow$ $|f(z) - f_n(z)| < \frac{\varepsilon}{3}$ $\forall z \in E$. Now choose $\delta > 0$ s.t. $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$ for $z \in E$, $|z - z_0| < \delta$. Then $z \in E$ and $|z - z_0| < \delta \Longrightarrow$
$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

so $f(z) \longrightarrow f(z_0)$ as $z \to z_0$, i.e. f cts on E. $\square$

Thm If $\gamma : I \to \mathbb{C}$ is a path, $\{f_n : \gamma(I) \to \mathbb{C}\} \xrightarrow[\text{unif}]{} f$, f_n cts, then $\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz$.

Pf Given $\varepsilon > 0$, choose N s.t. $n \geq N$, $z \in \gamma(I) \Longrightarrow |f(z) - f_n(z)| < \frac{\varepsilon}{\ell(\gamma)}$.
Then, for $n \geq N$,
$$\left| \int_\gamma f - \int_\gamma f_n \right| = \left| \int_\gamma f - f_n \right| \leq \frac{\varepsilon}{\ell(\gamma)} \ell(\gamma) = \varepsilon. \quad \square$$

e.g. $\{|z|^n\} \xrightarrow[\text{pointwise}]{} \begin{cases} z \mapsto 1 & \text{if } |z| = 1 \\ z \mapsto 0 & \text{if } |z| < 1 \end{cases}$ on $\overline{D}_1(0)$

Convergence is not uniform on $\overline{D}_1(0)$.

Defn Say that an infinite series $\sum_{k=0}^{\infty} f_k(z)$ of fns defined on E converges uniformly on E if the sequence of partial sums converges uniformly on E.

Thm [Weierstrass M-test] For $\sum_{k=0}^{\infty} f_k(z)$ as above, if there is a convergent series of nonnegative #s $\sum_{k=0}^{\infty} M_k$ s.t. $|f_k(z)| \leq M_k$ for all k and all $z \in E$, then $\sum_{k=0}^{\infty} f_k$ converges uniformly on E.

Pf Comparison test gives pointwise convergence to some fn s. Let $s_n = \sum_{k=0}^{n} f_k$. Then $|s(z) - s_n(z)| \leq \sum_{k=n+1}^{\infty} |f_k(z)| \leq \sum_{k=n+1}^{\infty} M_k$.

Since $\sum M_k$ converges, given $\varepsilon > 0$ may choose N s.t. $n \geq N \Rightarrow \sum_{k \geq n+1} M_k < \varepsilon \Longrightarrow |s(z) - s_n(z)| < \varepsilon$ for $n \geq N$, $z \in E$. ☐

e.g. $\left|\frac{z^k}{k^2}\right| \leq \frac{1}{k^2}$ for $|z| \leq 1$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ convergent implies $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges unif on $\overline{D}_1(0)$.

Radius of Convergence

Defn If $\{a_n\}$ is a sequence of real numbers, then $\limsup\{a_k\}$ is the limit of $\{u_n\}$ for $u_n = \sup\{a_k \mid k \geq n\}$.

Note $\{u_n\}$ is non-increasing, $\limsup\{a_n\}$ always well-defined in extended reals $[-\infty, \infty]$. $\{a_k\} \longrightarrow a \in [-\infty, \infty]$ iff $\limsup a_k = \liminf a_k = a$.

Thm Given a power series $\sum_{k=0}^{\infty} c_k (z-z_0)^k$, let

$$R = \frac{1}{\limsup |c_k|^{1/k}}.$$ Then the series converges absolutely on $D_R(z_0)$ and diverges on $\mathbb{C} - \overline{D}_R(z_0)$. Furthermore, unif conv on $\overline{D}_r(z_0)$ $\forall r < R$.

Defn R is the radius of convergence of the series.

Pf WLOG, $z_0 = 0$. Let $u_n = \sup\{|c_k|^{1/k} \mid k \geq n\}$ so that $\limsup |c_k|^{1/k} = \lim u_n$. If $r < R$, choose $r < t < R$. Then

$t^{-1} > R^{-1} = \lim u_n \implies$ for $n \gg 0$, $u_n < t^{-1}$

$\implies$ for $k \geq n$, $|c_k|^{1/k} < t^{-1}$

$\implies$ for $k \geq n$, $|c_k| < t^{-k}$

If $|z| \leq r$, this implies $|c_k z^k| < \left(\frac{r}{t}\right)^k$ for $k \geq n$.

Since $\frac{r}{t} < 1$, $\sum_{k=n}^{\infty} (r/t)^k$ converges. By Weierstrass M, $\sum_{k=n}^{\infty} c_k z^k$ converges uniformly on $\overline{D}_r(0)$, and the same is true for $\sum_{k=0}^{\infty} c_k z^k$ since unif conv unaffected by finitely many terms. Unif conv on $\overline{D}_r(0)$ for $r < R \implies$ absolut conv on $D_R(0)$.

Given $|z| > R$, $|z|^{-1} < \lim u_n \implies$ for each n there is $k > n$ with $|z|^{-1} < |c_k|^{1/k}$ so that $|c_k z^k| > 1$.

Thus $\{c_k z^k\} \not\to 0$ so the series diverges for $z \in \mathbb{C} - \overline{D}_R(0)$. $\square$

Cor Power series are continuous in their disc of convergence.

Prop If $\sum_{k \geq 0} c_k z^k$ has radius of conv R, then $\sum_{k \geq 1} c_k z^{k-1}$ has radius of conv R.

Pf Clearly $z \cdot \sum_{k\geq 1} k c_k z^{k-1} = \sum_{k\geq 0} k c_k z^k$ has the same radius of conv as $\sum_{k\geq 1} k c_k z^{k-1}$; call this R_1.

We have $R = \dfrac{1}{\limsup |c_k|^{1/k}}$

$$R_1 = \frac{1}{\limsup |k c_k|^{1/k}} = \frac{1}{\limsup k^{1/k} |c_k|^{1/k}}.$$

$\lim k^{1/k} = 1$, so $R_1 = R$. $\square$

Thm Let $f(z) = \sum_{k\geq 0} c_k (z-z_0)^k$ with radius of convergence R

Then $$\int_{z_0}^{z} f(w)\,dw = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(z-z_0)^{k+1} \quad \forall z \in D_R(z_0).$$

Pf Let s_n be the nth partial sum of f.

Then
$$\begin{array}{ccc} \displaystyle\int_{z_0}^{z} s_n(w)\,dw & = & \displaystyle\sum_{k=0}^{n} \frac{c_k}{k+1}(z-z_0)^{k+1} \\ \Big\downarrow \text{unif} & & \Big\downarrow \text{unif} \\ \displaystyle\int_{z_0}^{z} f(w)\,dw & & \text{RHS} \end{array}$$

$\square$

Power Series Expansion

Thm Let $f(z) = \sum_{n\geq 0} c_n (z-z_0)^n$ with radius of conv R.

Then f is analytic on $D_R(z_0)$ and $f'(z) = \sum_{n\geq 1} n c_n (z-z_0)^{n-1}$ with radius of conv R.

Pf Let $g(z) = \sum_{n\geq 1} n c_n (z-z_0)^{n-1}$. Previously saw that g ~~converges uniformly on $D_R(z_0)$~~ has radius of conv R.

Know g is cts on $D_R(z_0)$ and $\int_{z_0}^{z} g(w) = \sum_{n=1}^{\infty} c_n (z-z_0)^n = f(z) - f(z_0)$.

This is an antideriv of g and $f(z_0)$ is constant, so $f' = g$. □

e.g. Let $\text{Log} = \log_{(-\pi,\pi]}$ be the principal branch of the logarithm. We know $\text{Log}' = \frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$

Clearly $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$ has $\frac{1}{z}$ as its derivative as well. Thus $f' = \text{Log}'$ and $f = C + \text{Log}$.

But $f(1) = 0 = \text{Log}(1)$, so $f = \text{Log}$. □

Cor If f has a power series exp'n about z_0 w/ radius of conv R, then it has derivatives of all orders on $D_R(z_0)$. Its k-th derivative is $f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!\, c_n}{(n-k)!} (z-z_0)^{n-k}$

$$\text{and } f^{(k)}(z_0) = k!\, c_k. \quad \square$$

Cor If f has a power series exp'n about z_0 with positive radius of conv, then it has only one such expansion, and

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Power Series Expansions of Analytic Functions:

Thm Let f be analytic in an open set $U \subseteq \mathbb{C}$ and suppose $D_r(z_0) \subseteq U$ for some $r > 0$. Then there is a power series expansion for f, $f(z) = \sum_{n \geq 0} c_n (z-z_0)^n$, converging to $f(z)$ on $D_r(z_0)$.

Furthermore, $c_n = \frac{1}{2\pi i} \int_{|w-z_0|=s} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$,

where s is any number with $0 < s < r$.

Pf If $0 < t < s < r$, $|w - z_0| = s$, $|z - z_0| \leq t$, then

$$\left| \frac{z-z_0}{w-z_0} \right| \leq \frac{t}{s} < 1$$

so
$$\frac{w-z_0}{w-z} = \frac{1}{1 - \frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n \quad ⊛$$

with the final geometric series dominated by $\sum (t/s)^n$, which converges. By the M-test, ⊛ converges uniformly as a fn of $z \in D_t(z_0)$ and also of $w \in \partial D_s(z_0)$.

If we multiply ⊛ by $\frac{f(w)}{w-z_0}$ and integrate around $\partial D_s(z_0)$ to get

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_s(z_0)} \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\int_{\partial D_s(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\, dw \right) (z-z_0)^n$$

a power series exp'n for f on $D_t(z_0)$ with the appropriate coeffs. Since the coeffs don't depend on s, get conv on $D_r(z_0)$. $\square$

Cor If f is analytic on an open set U, then f has derivatives of all orders on U and they are all analytic. $\square$

Note: Power series exp'n is local, on the largest open disc in U centered at a given point

e.g. Log:

$$\sum (-1)^n \frac{(z-1)^n}{n}$$

Note: f analytic on open $U \supseteq \overline{D}_R(z_0)$. Then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\, dw .$$

Cor [Cauchy's Estimates] If f is analytic on open $U \supseteq \overline{D}_R(z_0)$, and if $|f(z)| \le M$ on $\partial D_R(z_0)$, then

$$|f^{(n)}(z_0)| \le \frac{n!\, M}{R^n}$$

for $n = 0, 1, 2, \dots$

Pf
$$\left| \frac{f(w)}{(w-z_0)^{n+1}} \right| \le \frac{M}{R^{n+1}} \quad \text{on } \partial D_R(z_0).$$

$$\Longrightarrow |f^{(n)}(z_0)| \le \frac{n!}{2\pi} \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M}{R}. \quad \square$$

Thm $\{f_n : U \subseteq \mathbb{C} \xrightarrow[\text{analytic}]{} \mathbb{C}\} \xrightarrow[\substack{\text{on compact} \\ K \subseteq U}]{\text{unif}} f$. Then f is analytic.

Pf Read pp. 87–88. $\square$

Liouville's Thm

Defn A function analytic on all of $\mathbb{C}$ is called *entire*.

Thm [Liouville] The only bounded entire functions are the constant functions.

Pf By the Cauchy estimates, if $|f(z)| \le M$ $\forall z \in D_R(z_0)$, then $|f'(z_0)| \le \frac{M}{R}$. If f is bounded by M on $\mathbb{C}$, then this holds for all $R>0$. Taking $R \to \infty$, get $f'(z_0)=0$. Hence f is constant. $\square$

Thm If f is defined and cts on $\mathbb{C}$ and $\lim_{z\to\infty} f(z)$ exists, then f is bounded on $\mathbb{C}$.

Here, $\lim_{z\to\infty} f(z) = L \in \mathbb{C}$ means $\forall \varepsilon>0$ $\exists R>0$ st $|z|>R$ $\Rightarrow |f(z)-L|<\varepsilon$.

Pf If $\lim_{z\to\infty} f(z) = L$, then $\exists R>0$ s.t. $|f(z)-L|<1$ for $|z|>R$.

Thus $|f(z)| < |L|+1$ if $|z|>R$. f cts on $\mathbb{C}$, $\overline{D}_R(0)$ compact $\Rightarrow f$ bdd on $\overline{D}_R(0)$ hence bdd on $\mathbb{C}$. $\square$

Fundamental Theorem of Algebra Every nonconstant complex polynomial has a complex root.

Pf Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a poly of deg $n \ge 1$ so $a_n \ne 0$. Assume for $\cancel{\text{C}}$ that p has no root in $\mathbb{C}$, so $p(z)\neq 0$ $\forall z\in\mathbb{C}$. Then $\frac{1}{p}$ is an entire function, and we will show it is also bounded. Let $h(z) = \frac{z^n}{p(z)} = \frac{1}{a_n + a_{n-1}z^{-1} + \cdots + a_1 z^{1-n} + a_0 z^{-n}}$.

Then $\dfrac{1}{p(z)} = \dfrac{h(z)}{z^n}$ for $z \neq 0$. Furthermore, $\lim_{z\to\infty} h(z) = 1/a_n$,

so $\lim_{z\to\infty} \dfrac{1}{p(z)} = \lim_{z\to\infty} \dfrac{h(z)}{z^n} = 0$. Thus $1/p$ is bounded on all of $\mathbb{C}$, so Liouville's Thm implies $1/p$ is constant, ⚡. $\square$

Cor Each complex polynomial factors completely into constant and monic linear factors.

Cor Every $A \in M_{n\times n}(\mathbb{C})$ has at least one cpx eigenvalue.

Cor We can solve a bunch of differential equations.

etc etc!

Thm An entire function f is a polynomial of degree $\leq n$ iff $\exists A, B > 0$ s.t. $|f(z)| \leq A + B|z|^n$ $\forall z \in \mathbb{C}$.

Pf Suppose $p(z) = a_n z^n + \cdots + a_0$. Then $\lim_{z\to\infty} \dfrac{p(z)}{z^n} = a_n$

so $\exists R > 0$ s.t. $|z| > R \Rightarrow \left|\dfrac{p(z)}{z^n} - a_n\right| < 1$

$$\Longrightarrow \frac{|p(z)|}{|z^n|} < |a_n| + 1$$

$$\Longrightarrow |p(z)| < \underbrace{(|a_n|+1)}_{B}|z|^n$$

Take $A > 0$ s.t. $|p(z)| \leq A$ on $\overline{D}_R(0)$ (by EVT).

Then $|p(z)| \leq A + B|z|^n$ on $\mathbb{C}$.

Converse: HW 6 via Cauchy's estimates. $\square$

Zeroes and Singularities

Thm If f is a function analytic on $U \subseteq \mathbb{C}$ open, then $\forall z_0 \in U$, exactly one of the following is true.

(a) there is an open disc $D_r(z_0)$ on which $f = 0$.

(b) there is a nonnegative integer k, open disc $D_r(z_0)$, and fn g, analytic on U, s.t.

$$f(z) = (z-z_0)^k g(z) \quad \forall z \in D_r(z_0)$$

and $g(z) \neq 0 \ \forall z \in D_r(z_0)$.

Pf Know $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ on $D_R(z_0)$ for some $R > 0$.

If all $c_n = 0$, then (a) holds. O/w call k smallest index s.t. $c_k \neq 0$. Then $f(z) = \sum_{n=k}^{\infty} c_n (z-z_0)^n = (z-z_0)^k \sum_{n=0}^{\infty} c_{n+k} (z-z_0)^n$.

Define
$$g(z) = \begin{cases} \sum_{n=0}^{\infty} c_{n+k} (z-z_0)^k & \text{if } z \in D_R(z_0) \\ \dfrac{f(z)}{(z-z_0)^k} & \text{if } z \in U \setminus \{z_0\}. \end{cases}$$

Now $g(z_0) = c_k \neq 0$ and g is cts, so $g(z) \neq 0$ on some $D_r(z_0)$, $0 < r < R$.

This gives (b). $\square$

Q What is the distribution of zeroes of an analytic fn?

Thm Suppose f analytic on a connected open $U \subseteq \mathbb{C}$ and f is not identically 0. Then

(a) for each $z_0 \in U$, $\exists k \in \mathbb{N}$, $r > 0$, $g : U \to \mathbb{C}$ analytic s.t.

$$f(z) = (z-z_0)^k g(z) \quad \forall z \in U$$

and $g(z) \neq 0 \ \forall z \in D_r(z_0)$

(b) $\forall z_0 \in U$ $\exists r > 0$ st f has no zeroes on $D_r(z_0)$ except possibly at z_0.

(c) The set of zeroes of f is at most countable.

Pf Let $V_1 = \{z_0 \in U \mid$ (a) from previous Thm holds$\}$

$V_2 = \{z_0 \in U \mid$ (b) from previous Thm holds$\}$.

Then V_1, V_2 open, $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = U$. Since U conn'd, one of $V_j = \emptyset$, the other is U. But $V_1 = U$ contradicts $f \not\equiv 0$, thus $V_2 = U$, and this proves (a) & (b) of this thm.

To prove (c), modify discs of (b) s.t. centers are in $\mathbb{Q}(i)$, radii $\in \mathbb{Q}_{>0}$, z_0 still only 0 of f in the disc.

$|\mathbb{Q}^3| = \aleph_0$. $\square$

Note For $f: \underset{\text{open}}{U \subseteq \mathbb{C}} \to \mathbb{C}$, let $Z(f) = \{z \in U \mid f(z) = 0\}$.

Then $Z(f)$ consists of isolated points.

Defn For $E \subseteq \underset{\text{open}}{U} \subseteq \mathbb{C}$, call E a discrete subset of U if every point z_0 of U has a nbhd containing no points of E except possibly z_0 itself.

Then (b) of Thm says that for $f: \underset{\text{open conn'd}}{U \subseteq \mathbb{C}} \to \mathbb{C}$ analytic, $Z(f)$ is a discrete subset of U.

discrete   isolated, not discrete

Thm $f, g: U \to \mathbb{C}$ analytic on open conn'd $U \subseteq \mathbb{C}$. If $f(z) = g(z)$ $\forall z$ in a nondiscrete subset E of U, then $f = g$.

Pf Let $h = f - g$. Then h analytic, $= 0$ on a nondiscrete subset of U, so $h = 0$ on U. $\square$

e.g. $\cos z - 1 = \frac{-z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$

$$= z^2 \underbrace{\left(-\frac{1}{2!} + \frac{z^2}{4!} - \cdots + (-1)^n \frac{z^{2n-2}}{(2n)!} + \cdots\right)}_{g(z)}$$

order 2 zero at 0

Thm (a) If an analytic fn g is not 0 at z_0 in its domain, then in some nbhd V of z_0 there is an analytic fn $h: V \to \mathbb{C}$ s.t. $g(z) = e^{h(z)}$. (b) If z_0 is a zero of order k for f, then $f(z) = (z-z_0)^k e^{h(z)}$ for some analytic h on nbhd V of z_0.

Pf (a) Choose a branch of log that does not have $g(z_0)$ on its cut line. Let W be its domain. Set $V = g^{-1}(W)$ and $h(z) = \log(g(z))$. Then $h(z) = \log(g(z))$. $\square$

Isolated Singularities

Defn If $U \subseteq \mathbb{C}$ open, $z_0 \in U$, f analytic on $U \setminus \{z_0\}$ but not on U, say f has an isolated singularity at z_0. If f can be given a value at z_0 so that it becomes analytic on U, call the singularity removable.

Thm If f has an isolated singularity at z_0 and is bounded in some deleted nbhd of z_0, then z_0 is a removable singularity of f.

Pf Suppose f is analytic and bounded on $U \setminus \{z_0\}$. Define g by $g(z) = (z-z_0)^2 f(z)$ for $z \neq z_0$, and $g(z_0) = 0$. Then

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z-z_0) f(z) = 0 \quad \text{(b/c } f \text{ is bounded)}$$

Thus g is analytic on U. Since $g(z_0) = g'(z_0) = 0$, know that the first two terms in its power series about z_0 are 0. So factor out $(z-z_0)^2$ to get $g(z) = (z-z_0)^2 h(z)$ for h analytic, defined by power series at z_0. In a sufficiently small disc,

$$g(z) = (z-z_0)^2 f(z) = (z-z_0)^2 h(z) \implies f = h \text{ in this disc. } \square$$

e.g. $f(z) = \dfrac{\cos z - 1}{z^2}$ has a removable singularity at 0.

Defn A fn $f: U \setminus \{z_0\} \to \mathbb{C}$ of the form

$$f(z) = \frac{g(z)}{(z-z_0)^k}$$

with g analytic on U, $g(z_0) \neq 0$, $k \in \mathbb{Z}^+$, is said to have a pole of order k at z_0. If $k=1$, call this a simple pole. An isolated singularity which is not removable and not a pole is called essential.

e.g. $f(z) = \dfrac{1}{1-e^z}$ has simple poles at $\{2\pi k i \mid k \in \mathbb{Z}\}$.

Indeed, $(1-e^z)' = -e^z \neq 0$.

Thm If f is analytic on $U \setminus \{z_0\}$ and has an essential singularity at z_0, then for every open disc D centered at z_0 and contained in U, $f(D \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Pf Read p. 99. $\square$

Moral Essential singularities are wild! Cf. Big & little Picard

Thm Let f be an analytic fn with isolated singularity at z_0.
Then (a) f has removable sing. at z_0 iff $\lim_{z \to z_0} f(z) \in \mathbb{C}$.

(b) f has a pole at z_0 iff $\lim_{z \to z_0} f(z) = \infty$

(c) f has an essential sing at z_0 iff $\lim_{z \to z_0} f(z)$ DNE. $\square$

e.g. $f(z) = e^{1/z}$ has an essential sing at 0.

$$f\left(\frac{1}{2\pi n i}\right) = e^{2\pi n i} = 1 \quad \forall n \in \mathbb{Z}$$

$$f\left(\frac{1}{n}\right) = e^{n} \quad \forall n \in \mathbb{Z}.$$

Meromorphic Functions

Defn Let $U \subseteq \mathbb{C}$ open, $E \subseteq U$ discrete. If f is analytic on $U - E$ and has a removable sing or pole at all points of E, then f is called meromorphic on U.

Thm If U is a connected open set and f is a meromorphic fn on U, $f \not\equiv 0$, then $1/f$ is meromorphic.

Pf Know $Z(f)$ = zeroes of f is discrete, and $P(f)$ = poles of f is discrete. Thus $E = Z(f) \cup P(f)$ is discrete. The fn $1/f$ is analytic on $U - E$. For $z_0 \in E$ $\exists D = D_r(z_0)$ where f analytic on $D - \{z_0\}$, f has a zero or pole at z_0.

If f has a zero of order k at z_0, then $f(z) = (z-z_0)^k g(z)$ for g analytic on D, $g(z_0) \neq 0$. Thus $\frac{1}{f(z)} = \frac{1/g(z)}{(z-z_0)^k}$ has a pole of order k at z_0.

If f has pole of order k at z_0, then $1/f$ has a zero of order k at z_0.

$\square$

Examples, Examples, Examples

Note that when f has a pole of order k at z_0, then

$$f(z) = \frac{g(z)}{(z-z_0)^k} \quad \text{for } g \text{ analytic in } D_r(z_0),\ g(z_0) \neq 0.$$

But then g has a power series expansion $g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ with $c_0 \neq 0$. Hence

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^{n-k} \quad \text{on } D_r(z_0) \setminus \{z_0\}.$$

This is a __Laurent series__ expansion of f.

Generally write these as $\sum_{n=-N}^{\infty} a_n (z-z_0)^n$.

pole of order N if $a_{-N} \neq 0$.

__e.g.__ Let's find a Laurent series for $\frac{z}{z^2+1}$ at $z_0 = i$.

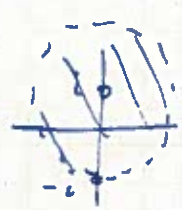
$$\frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)} = \frac{1}{2}\,\frac{1}{z-i} + \frac{1}{2}\,\frac{1}{z+i}.$$

$\frac{1}{z+i}$ analytic at i with power series

$$\frac{1}{z+i} = \frac{1}{2i + (z-i)} = \frac{1}{2i}\,\frac{1}{1-\left(-\frac{z-i}{2i}\right)}$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}\left(-\frac{z-i}{2i}\right)^n = \sum_{n=0}^{\infty} i^{-n-1} 2^{-n-1} (z-i)^n.$$

converging on $D_2(i)$



Thus $$\frac{z}{z^2+1} = \frac{1}{2}(z-i)^{-1} + \sum_{n=0}^{\infty} i^{n-1} 2^{-n-2} (z-i)^n.$$

e.g. What is the power series of $\frac{e^z}{1-z}$ at $z_0 = 0$?

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \text{conv on } \mathbb{C}$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \quad \text{conv on } D_1(0).$$

<u>Hope</u>
$$\frac{e^z}{1-z} = (1+z+z^2+z^3)\cdot\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$$

$$= 1 + (z+z) + \left(\frac{z^2}{2} + z^2 + z^2\right) + \left(\frac{z^3}{6} + \frac{z^3}{2} + z^3 + z^3\right) + \cdots$$

$$= 1 + 2z + \frac{5z^2}{2} + \frac{8z^3}{3} + \cdots \quad \text{for } |z| < 1.$$

<u>Lemma</u> Suppose $\sum a_n z^n$ and $\sum b_n z^n$ converge for $|z| < r_0$. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum c_n z^n$ converges for $|z| < r_0$ and

$$\sum c_n z^n = \left(\sum a_n z^n\right)\left(\sum b_n z^n\right).$$

<u>Pf</u> Let $f(z) = \sum a_n z^n$, $g(z) = \sum b_n z^n$. Both f, g analytic on $D_{r_0}(0)$, so $f \cdot g$ is analytic on $D_{r_0}(z_0)$. Thus

$$(fg)(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} z^n$$

Furthermore, $(fg)^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)$.

Hence
$$\frac{(fg)^{(n)}(0)}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} f^{(k)}(0)\, g^{(n-k)}(0)$$

$$= \sum_{k=0}^{n} a_k b_{n-k} = c_n. \quad \square$$

In particular, our Hope is true.

e.g. Find f s.t. $f(0)=0$ and $f'(x) = 3x+2$.

Suppose f exists and is analytic, so $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Since $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, we must have

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = 3\left(\sum_{n=0}^{\infty} a_n z^n\right) + 2$$

i.e. $$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n = (2+3a_0) + \sum_{n=1}^{\infty} 3a_n z^n$$

i.e. $$0 = (2+3a_0-a_1) + \sum_{n=1}^{\infty} \left(3a_n - (n+1)a_{n+1}\right) z^n.$$

Now $a_0 = f(0) = 0 \Rightarrow a_1 = 2$. For $n \geq 1$, $a_{n+1} = \frac{3a_n}{n+1}$

$$\Rightarrow a_2 = \frac{3a_1}{2}, \quad a_3 = \frac{3^2 a_1}{3\cdot 2}, \quad a_4 = \frac{3^3 a_1}{4\cdot 3\cdot 2}, \ldots$$

and $$a_n = \frac{3^{n-1} a_1}{n!} = 3^n \left(\frac{2}{3}\right) n!$$

Thus any power series sol'n is necessarily of the form

$$f(z) = \frac{2}{3} \sum_{n=1}^{\infty} \frac{3^n}{n!} z^n$$

$$= \frac{2}{3}\left(e^{3z} - 1\right) \checkmark$$

TPS Find Laurent series for

$$\frac{\cos z}{z^2}, \quad \frac{e^z - 1}{z^2}, \quad \frac{z+1}{z-1}$$

at $z_0 = 0$.

Maximum Modulus Principle

Thm If f is analytic on a conn'd open set $\mathcal{U} \subseteq \mathbb{C}$ and $|f|$ has a local max at $z_0 \in \mathcal{U}$, then f is constant on $\mathcal{U}$.

Lemma Let $f: I = [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ be cts. If

$$|f(t)| \leq M := \left| \frac{1}{b-a} \int_a^b f(t)\, dt \right|, \quad \forall t \in I$$

then f has constant modulus M on I.

Pf Lemma Choose $u \in \mathbb{C}$, $|u| = 1$ s.t. $u \int_a^b f(t)\, dt = \left| \int_a^b f \right|$.

Then $\int_a^b (M - uf(t))\, dt = 0$. Let $uf = g + ih$, $g, h : I \to \mathbb{R}$.

Then $|f(t)| \leq M \Rightarrow g(t) \leq M \Rightarrow M - g(t) \geq 0$.

Have $\int_a^b (M - g(t))\, dt = 0$ and $\int_a^x (M - g(t))\, dt$ diff'l in x

w/ derivative $M - g(x) \geq 0 \Rightarrow$ non-decreasing fn.

Since 0 at $x = a$, $x = b$, must be constant $\Rightarrow M = g(t)$.

Thus $uf = M + ih$ and

$$M^2 \geq |f(t)|^2 = |uf(t)|^2 = g(t)^2 + h(t)^2 = M^2 + h(t)^2$$

$\Rightarrow h(t) = 0 \ \forall t \in I$. Thus $f = u^{-1} M$ which has modulus M. $\square$

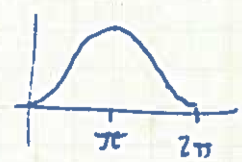
Pf Thm Choose $r > 0$ s.t. $\overline{D}_r(z_0) \subseteq \mathcal{U}$ and $|f(z_0)|$ max for $|f(z)|$ on $\overline{D}_R(z_0)$. By Cauchy's integral thm,

$$f(z_0) = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Since $|f(z_0 + re^{it})| \leq |f(z_0)|$ on $[0, 2\pi]$, may apply the lemma with $M = |f(z_0)|$. It follows that f is constant on $z_0 + re^{it}$, a non-discrete subset of U. By the identity thm, f is constant on all of U. $\square$

<u>Cor</u> Suppose U conn'd, bdd, open $\subseteq \mathbb{C}$. If f is cts on $\overline{U}$, analytic on U, and nonconstant, then $\max_{\overline{U}} |f(z)|$ is attained on ∂U and nowhere else.

<u>Pf</u> $|f|$ attains a max by EVT applied to $\overline{U}$. By the Thm, $|f|$ has no local max on U, so must on $\overline{U} - U = \partial U$. $\square$

<u>e.g.</u> Where does $f(z) = z^2 - z$ attain max modulus on $\overline{D}_1(0)$?
By Cor, on $S^1 = \{e^{it} \mid t \in [0, 2\pi]\}$, so only must maximize
$$h(t) = |e^{2it} - e^{it}|^2 = |e^{it} - 1|^2 = 2 - 2\cos t.$$


This is clearly at $t = \pi$, so max modulus of f is $|f(-1)| = 2$.

<u>Schwarz's Lemma</u> Let f be analytic on $D_1(0)$ w/ $f(0) = 0$ and $|f(z)| \le 1$ for every $z \in D_1(0)$. Then $|f(z)| \le |z|$ for all $z \in D_1(0)$ and $|f'(0)| \le 1$. If $|f'(0)| = 1$, then $f(z) = cz$ for some constant $c \in \mathbb{C}$.

<u>Pf</u> Since $f(0) = 0$, $f(z) = z g(z)$ with g analytic on $D_1(0)$.
Since $|f(z)| \le 1$, $|g(z)| \le \frac{1}{r}$ on $|z| = r$, for each $r < 1$.
By Max Modulus Thm, this also holds for $|z| < r$. Thus $|g(z)| \le 1$ on $D_1(0)$. Hence $|f(z)| = |z||g(z)| \le |z|$.
Now $f'(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} g(z) = g(0)$, so $|f'(0)| \le 1$.
If $|f'(0)| = 1$, then $|g(0)| = 1$ is max modulus of g on $D_1(0)$
$\Rightarrow$ g constant. $\square$

Defn Let $U, V \subseteq \mathbb{C}$ open. A bi-analytic map from U to V is an analytic fn $f: U \to V$ with an analytic inverse $f^{-1}: V \to U$.

Thm The only bi-analytic maps $D_1(0) \to D_1(0)$ that take 0 to 0 are of the form $f(z) = cz$ for $|c| = 1$. I.e., just rotations.

Pf Both f, f^{-1} satisfy Schwarz's lemma, so $|f'(0)| \le 1$ and $|(f^{-1})'(0)| \le 1$. Applying the chain rule to $f^{-1} \circ f = id$, we have $(f^{-1})'(0) = \frac{1}{f'(0)} \implies |f'(0)| = 1$, and the conclusion follows from SL. □

Harmonic Functions

Thm Let u be a function of class C^2 harmonic on a convex open set U. Then u has a harmonic conjugate v on U.

Pf Let $g = u_x - iu_y$, which is C^1 and

$$u_{xx} = -u_{yy}, \quad u_{xy} = u_{yx}$$

so g is analytic on U. Since U is convex, g has an ~~antideriv~~ primitive h on U, h analytic w/ $h' = g$. If $h = w + iv$, then $u_x - iu_y = g = h' = w_x + iv_x = w_x - iw_y$

$$\implies u_x = w_x, \quad u_y = w_y.$$

Thus $w = u + c$, $c \in \mathbb{R} \implies f = h - c = u + iv$ analytic w/ $\mathrm{Re}(f) = u$.

□

Thm If u is harmonic on conn'd open U and u has a local max at some $z_0 \in U$, then u is constant on U.

Pf p. 105 □

<u>Thm</u> If u is harmonic on $U \subseteq \mathbb{C}$ open, $\bar{D}_r(z_0) \subseteq U$, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\,dt .$$

Pf Cauchy integral thm. $\square$

Chains and Cycles

Defn For $U \subseteq \mathbb{C}$ open, a 1-chain on U is a formal $\mathbb{Z}$-linear combination of paths $\gamma_i : [0,1] \to U$,

$$\Gamma = \sum_{i=1}^{p} m_i \gamma_i$$

where γ_i are distinct and $0 \cdot \gamma = 0$. These form an Abelian group under addition:

$$\sum_{i=1}^{p} m_i \gamma_i + \sum_{i=1}^{p} n_i \gamma_i = \sum_{i=1}^{p} (m_i + n_i) \gamma_i .$$

Note · Paths are 1-chains

· Every path is a sum of smooth paths

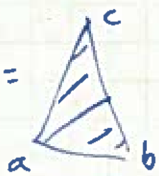
Defn For $\Gamma = \sum_{i=1}^{p} m_i \gamma_i$, $I = [0,1]$, set $\Gamma(I) = \bigcup_{m_i \neq 0} \gamma_i(I)$.

If f is cts on $E \subseteq \mathbb{C}$ with $\Gamma(I) \subseteq E$, define

$$\int_\Gamma f = \sum_{i=1}^{p} m_i \int_{\gamma_i} f .$$

Prop $\int_{\Gamma + \Lambda} f = \int_\Gamma f + \int_\Lambda f$, $\int_{m\Gamma} f = m \int_\Gamma f \quad \forall m \in \mathbb{Z}$. $\square$

Defn Suppose Γ, Λ are 1-chains with $\Gamma(I), \Lambda(I) \subseteq E \subseteq \mathbb{C}$. Call Γ, Λ E-equivalent if $\int_\Gamma f = \int_\Lambda f \quad \forall$ cts f on E.

e.g. $\Delta =$ 

then $\partial\Delta$ & $[a,b] + [b,c] + [c,a]$ & $[a,b] + [b,c] - [a$
are equivalent.

Defn A 0-chain in U is a $\mathbb{Z}$-linear combination of singleton subsets of $\mathbb{C}$, $\sum_{i=1}^{p} m_i \{z_i\}$, $m_i \in \mathbb{Z}$, $z_i \in \mathbb{C}$.

$$\partial\left(\sum_{i=1}^{p} m_i \gamma_i\right) = \sum_{i=1}^{p} \left(m_i \{\gamma_i(1)\} - m_i \{\gamma_i(0)\}\right)$$

(combine any like terms)

Note $\partial(\Gamma+\Lambda) = \partial(\Gamma) + \partial(\Lambda)$ so ∂ is a group homomorphism (it's $\mathbb{Z}$-linear)

Defn A 1-chain Γ in U is a cycle if $\partial\Gamma = 0$.

Thm If Γ is a 1-cycle, then there is a 1-cycle Λ equivalent to Γ which is a sum of closed paths.

Pf We make changes to Γ which don't change integrals over it or $\Gamma(I)$: First write Γ as a sum of paths w/ coeff 1:

$$m\gamma \to \gamma + \gamma + \cdots + \gamma \quad \text{if } m>0$$
$$\to (-\gamma) + (-\gamma) + \cdots + (-\gamma) \quad \text{if } m<0$$

This results in $\tilde{\Gamma}$, a sum of n paths. If not all paths closed, have γ_j in $\tilde{\Gamma}$ with $\gamma_j(0) \neq \gamma_j(1)$. Since $\partial\tilde{\Gamma} = 0$, know $\gamma_j(1) = \gamma_k(0)$ for some term γ_k of $\tilde{\Gamma}$. Join γ_j and γ_k to express $\tilde{\Gamma}$ as $\tilde{\tilde{\Gamma}}$ with $n-1$ terms.

Proceeding by induction, get a sum of closed paths. $\square$

Index of a cycle

Defn If Γ is a 1-cycle and $z \in \mathbb{C} \setminus \Gamma(I)$, define

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w-z}.$$

the index of Γ around z.

Thm Γ a 1-cycle in $\mathbb{C}$. Then

(a) $\mathrm{Ind}_\Gamma : \mathbb{C} \setminus \Gamma(I) \to \mathbb{Z}$

(b) Ind_Γ is locally constant

(c) Ind_Γ is 0 on the unbdd cpt of $\mathbb{C} \setminus \Gamma(I)$.

(d) If Λ is a cycle, $z \in \mathbb{C} \setminus (\Gamma(I) \cup \Lambda(I))$, then

$$\mathrm{Ind}_{\Gamma+\Lambda}(z) = \mathrm{Ind}_\Gamma(z) + \mathrm{Ind}_\Lambda(z).$$

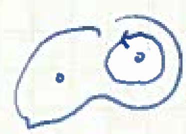
e.g. $\gamma(t) = 2e^{2\pi i t}$, $\lambda(t) = e^{2\pi i t}$, $t \in [0,1]$ then $\mathrm{Ind}_{\gamma-\lambda}(z) = 0$ everywhere

Homologous Cycles

Defn $U \subseteq \mathbb{C}$ open, Γ, Λ 1-cycles in U are homologous in U if
$$Ind_\Gamma(z) = Ind_\Lambda(z)$$
for all z in $\mathbb{C} \setminus U$. Call Γ homologous to 0 in U if $Ind_\Gamma(z) = 0$ $\forall z \in \mathbb{C} \setminus U$.

Intuition: Components of Γ don't "go around any holes in U"

homologous to 0



not homologous to 0.

Note Γ homologous to Λ iff $\Gamma - \Lambda$ homologous to 0.

Cauchy's Theorems

For f analytic on $U \subseteq \mathbb{C}$ open, define
$$g(z,w) = \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases},$$
a well-defined function $g: U \times U \to \mathbb{C}$.

Lemma g is continuous.

Pf Clearly cts for $w \neq z$. Need to show
$$\lim_{(z,w)\to(z_0,z_0)} g(z,w) = f'(z_0).$$

If $z \neq w$, $f(w)-f(z) = \int_z^w f'(\lambda)\,d\lambda$ so
$$|g(z,w) - f'(z_0)| = \Big| \frac{1}{w-z} \int_z^w \underbrace{(f'(\lambda) - f'(z_0))}_{\text{small by continuity of } f'} d\lambda \Big|$$

If $z = w$, then $|g(z,w) - f'(z_0)| = |f'(z) - f'(z_0)|$ is again small. $\square$

Cauchy's Integral Formula Let $U \subseteq \mathbb{C}$ open, f analytic on U, Γ a 1-cycle in U homologous to 0 in U. Then
$$\mathrm{Ind}_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\,dw \quad \forall z \in U - \Gamma[I].$$

Pf Let $h(z) = \int_\Gamma g(z,w)\,dw$, which is cts by Lemma
$$= \int_\Gamma \frac{f(w)}{w-z}\,dw - \int_\Gamma \frac{f(z)}{w-z}\,dw = \int_\Gamma \frac{f(w)}{w-z}\,dw - 2\pi i\,\mathrm{Ind}_\Gamma(z) f(z)$$
Thus need to show $h(z) \equiv 0$ on U. To do so, we prove h is entire, bounded, with $\lim_{z\to\infty} h(z) = 0$.

For Δ a triangle in $\mathcal{U}$,

$$\int_{\partial\Delta} h(z)\,dz = \int_{\partial\Delta}\int_\Gamma g(z,w)\,dw\,dz$$

$$= \int_\Gamma \underbrace{\int_{\partial\Delta} g(z,w)\,dz}\,dw \qquad \text{[Fubini]}$$

for fixed w, analytic for $z \neq w$, so Cauchy's Thm on $\partial\Delta$:

$$= \int_\Gamma 0\,dw = 0.$$

By Morera's Thm, h is analytic on $\mathcal{U}$.

Let $V = \{z \in \mathbb{C} \setminus \Gamma(I) \mid \mathrm{Ind}_\Gamma(z) = 0\} \subseteq \mathbb{C}$ open.

Then $V \supseteq \mathbb{C} \setminus \mathcal{U}$ by hypothesis, hence $\mathcal{U} \cup V = \mathbb{C}$.

If $z \in V \cap \mathcal{U}$, then $\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{\cancel{w-z}}\,dw = f(z)\,\mathrm{Ind}_\Gamma(z) = 0$.

Thus $h(z) = \int_\Gamma \frac{f(w)}{w-z}\,dw - \int_\Gamma \frac{f(z)}{w-z}\,dw = \int_\Gamma \frac{f(w)}{w-z}\,dw$ for $z \in \mathcal{U} \cap V$.

Thus extend h to $\mathbb{C}$ by defining it to be $\int_\Gamma \frac{f(w)}{w-z}\,dw$ on V.

On the unbounded component of $\mathbb{C} \setminus \Gamma(I)$, h is given by

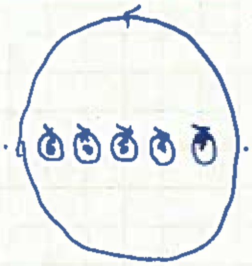
$$h(z) = \int_\Gamma \frac{f(w)}{w-z}\,dw \longrightarrow 0 \quad \text{as } z \to \infty.$$

By Liouville's Thm, $h \equiv 0$. $\square$

<u>Thm</u> [Cauchy] If f is analytic on $\mathcal{U} \subseteq \mathbb{C}$ open and Γ is a 1-cycle in $\mathcal{U}$ homologous to 0 on $\mathcal{U}$, then $\int_\Gamma f(z)\,dz = 0$.

<u>Pf</u> Fix $z_0 \in \mathcal{U} \setminus \Gamma(I)$ and define $g(z) = f(z)(z - z_0)$. Then g is analytic on $\mathcal{U}$ with $g(z_0) = 0$. Thus $\int_\Gamma f(z)\,dz = \int_\Gamma \frac{g(z)}{z - z_0}\,dz$
$= 2\pi i\,\mathrm{Ind}_\Gamma(z_0)\,g(z_0) = 0$. $\square$

e.g. Γ



Γ is homologous to 0 in $\mathbb{C} \setminus \mathbb{Z}$

Thus $\displaystyle\int_\Gamma \frac{1}{\sin(\pi z)}\, dz = 0$.

Simple Closed Paths

Defn A closed curve $\gamma : [a,b] \to \mathbb{C}$ is simple if $\gamma(s) \neq \gamma(t)$ for $a \le s < t \le b$ unless $s = a$ and $t = b$.

A simple closed path is a simple closed curve which is piecewise smooth with non-zero left and right derivatives at each pt.

(Weak) Jordan Curve Thm If γ is a simple closed path, then $\mathbb{C} \setminus \gamma(I)$ has exactly two components: a bounded component on which $\mathrm{Ind}_\gamma(z) = \pm 1$, and an unbounded component on which $\mathrm{Ind}_\gamma(z) = 0$.

Pf Choose for each $z \in \gamma(I)$ an open disc D_z centered at z with $D_z \setminus \gamma(I)$ consisting of components L_z, R_z with $\mathrm{Ind}_\gamma(z)$ one unit greater on L_z than on R_z. Sufficiently close z have overlapping D_z with overlapping L's and overlapping R's.

$L := \bigcup_{z \in \gamma(I)} L_z$, $R := \bigcup_{z \in \gamma(I)} R_z$ connected open sets, and

$$\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_\gamma(w) + 1 \quad \text{for } z \in L, w \in R.$$

Thus $L \cap R = \emptyset$. Let $U := \bigcup D_z = L \cup R \cup \gamma(I)$

Every component of $\mathbb{C} \setminus \gamma(I)$ has a subset of $\gamma(I)$ as its boundary. Thus every such component has nonempty intersection with U, hence meets L or R. By connectedness, only meets 1. Thus there are only two cpts, one containing L, the other R. One is unbounded with $\mathrm{Ind}_\gamma = 0$ on it. $\square$

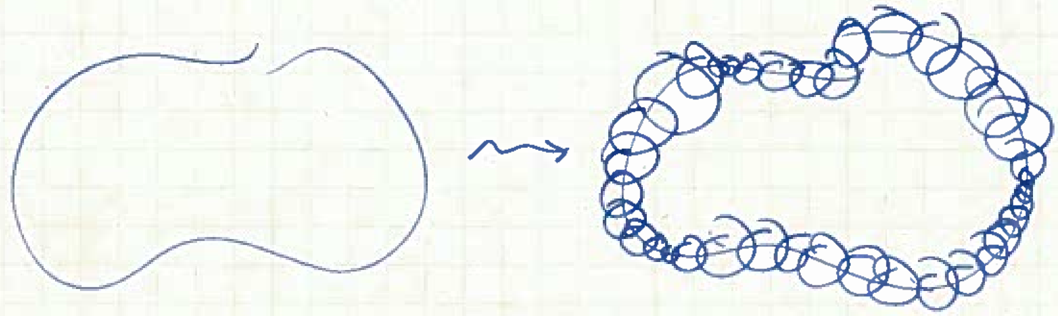
<u>Thm</u> If γ is a simple closed path and f is analytic on $U \subseteq \mathbb{C}$ open containing $\gamma(I)$ and its inside, then

$$\int_\gamma f(w)\,dw = 0$$

and

$$f(z) = \frac{\pm 1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

for each z on the inside of $\gamma(I)$. $\square$



Laurent Series

<u>Defn</u>. A <u>neighborhood of ∞</u> is any open set in $\mathbb{C}$ containing the complement of a closed bdd disc.

- If f is analytic in a nbhd of ∞, say it <u>vanishes at ∞</u> if $\lim_{z\to\infty} f(z) = 0$.

<u>Lemma</u> If h is analytic on $\mathbb{C} \setminus \overline{D}_r(z_0)$ and vanishes at ∞, then

$$q(w) = \begin{cases} h(1/w + z_0) & \text{if } w \neq 0 \\ 0 & \text{if } w = 0 \end{cases}$$

is analytic on $D_{1/r}(0)$.

<u>Pf</u> $\frac{1}{w} + z_0 \in \mathbb{C} \setminus \overline{D}_r(z_0)$ iff $\frac{1}{|w|} > r$ iff $w \in D_{1/r}(0)$.

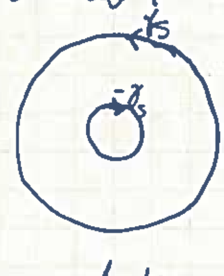
Clearly q is analytic on $D_{1/r}(0) \setminus \{0\}$ and $\lim_{w\to 0} q(w) = 0$ so q is analytic on $D_{1/r}(0)$. $\square$

<u>Defn</u> An open <u>annulus</u> centered at z_0 is a set of the form

$$A = \{ z \in \mathbb{C} \mid r < |z - z_0| < R \}$$

where $0 \le r < R \le \infty$.

For $r < s < S < R$ consider γ_s, γ_S in A pos around $|w - z_0| = s$, $|w - z_0| = S$.
Define $\Gamma = \gamma_S - \gamma_s$. Then



$$\mathrm{Ind}_\Gamma(z) = \begin{cases} 0 & \text{if } |z - z_0| > S \\ 1 & \text{if } s < |z - z_0| < S \\ 0 & \text{if } |z - z_0| < s. \end{cases}$$

Hence Γ nullhomologous in A, so if f is analytic on A, then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw = \begin{cases} 0 \\ f(z) \\ 0 \end{cases}$$

Thus if s, S are on the same side of $|z-z_0|$,

$$\frac{1}{2\pi i}\int_{\gamma_S} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w-z}\,dw.$$

If s, S are on opp sides of $|z-z_0|$,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_S} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w-z}\,dw.$$

Thm If f is analytic on an annulus A then $\exists!$ way to write

$$f(z) = g(z) - h(z) \quad \text{for } z \in A$$

where g is analytic on $D_R(z_0)$, and h is analytic on $\mathbb{C} \setminus \bar{D}_r(z_0)$ and vanishes at ∞.

Pf Define g as follows: if $|z-z_0| < R$, choose S with $|z-z_0| < S$ and $r < S < R$. Set $g(z) = \frac{1}{2\pi i}\int_{\gamma_S} \frac{f(w)}{w-z}\,dw$. By above, this doesn't depend on choice of S with $|z-z_0| < S < R$.

Define h on $\mathbb{C} \setminus \bar{D}_r(z_0)$ as follows: for $|z-z_0| > r$, choose s s.t. $r < s < R$, $s < |z-z_0|$. Set

$$h(z) = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w-z}\,dw$$

which doesn't depend on s.

If $z \in A$, $f(z) = g(z) - h(z)$ by prior work. Use Morera to get that g, h are analytic in appropriate domains

- $h(z) \longrightarrow 0$ as $z \longrightarrow \infty$: simple check.
- uniqueness : Identity Thm $\square$

Laurent Series Expansion

Thm If f is analytic on annulus A, then f has a unique exp'n of the form $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ which converges to f at all pts of A and converges unif on cpt subsets of A.

Pf Write $f = g - h$ as in previous thm. Then

$$g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \quad \text{on } D_R(0)$$

and $q(w)$ as in Lemma is analytic on $D_{1/r}(0)$ with $q(0)=0$

$$\Rightarrow \quad q(w) = \sum_{n=1}^{\infty} b_n w^n = h(1/w + z_0)$$

Subbing $z = \frac{1}{w} + z_0$ (i.e. $w = (z-z_0)^{-1}$) get

$$h(z) = \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

Set $c_n = -b_n$ for $n<0$ to get $h(z) = -\sum_{n=-\infty}^{-1} c_n (z-z_0)^{n}$.

Then $f = g - h$ on A gives $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$. $\square$

e.g. $f(z) = \dfrac{1}{(z-1)(z-2)}$ on $A = \{1 < |z| < 2\}$

If $g(z) = \frac{1}{z-2}$ for $|z|<2$ and $h(z) = \frac{1}{z-1}$ for $|z|>1$, then

$f = g - h$ on A and $h(z) \to 0$ at ∞.

$$g(z) = \frac{-1/2}{1 - z/2} = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} \quad \text{in } D_2(0)$$

$$h(z) = \frac{1}{z-1} = \frac{1/z}{1 - 1/z} = \sum_{n=1}^{\infty} \frac{1}{z^n} \quad \text{on } \mathbb{C} \setminus \overline{D}_1(0). \quad \text{Thus}$$

$$f(z) = \sum_{n=-\infty}^{-1} (-1) z^n + \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} z^n \quad \text{on } A.$$

Recall On $D_1(0) \setminus \{0\}$, $f'(z) = -\sum_{n=-1}^{\infty} (z-1)^n$.

e.g. $f(z) = e^{1/z}$ analytic on $\mathbb{C} \setminus \{0\}$, $f(z) \to 1$ as $z \to \infty$ so

$g(z) = 1$, $h(z) = 1 - e^{1/z}$, $f(z) = \sum_{n=-\infty}^{0} \frac{z^n}{|n|!}$

Thm [Integral fmla for Laurent series coeffs] If $A = \{r < |z - z_0| < R\}$, f analytic on A, $r<s<R$, then the Laurent series of f on A has coeffs

$$c_k = \frac{1}{2\pi i} \int_{|w-z_0|=s} \frac{f(w)}{(w-z_0)^{k+1}}\, dw.$$

Pf We have $\frac{1}{2\pi i}\int_{|w-z_0|=s} \frac{f(w)}{(w-z_0)^{k+1}}\,dw = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{|w-z_0|=s} (w-z_0)^{n-k-1}\,dw}_{\int_0^{2\pi} i s^{n-k} e^{i(n-k)t}\,dt}$

$$= \begin{cases} 0 & \text{if } n \neq k \\ 2\pi i & \text{if } n = k \end{cases}. \quad \square$$

The Residue Thm

$D_r(z_0) \setminus \{z_0\}$ is an annulus so when f is analytic on $D_r(z_0)\setminus\{z_0\}$ with isolated singularity at z_0, then $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$

Defn The coefficient of $(z-z_0)^{-1}$ is the residue of f at z_0.

$$\operatorname{Res}(f, z_0) := c_{-1}.$$

$\boxed{\text{e.g. } \operatorname{Res}\left(\frac{g(z)}{z-z_0}, z_0\right) = g(z_0) \text{ for } g \text{ analytic on open} \ni z_0.}$

Thm For f analytic on $D_R(z_0)\setminus\{z_0\}$ with isolated sing at z_0 and $0<r<R$,

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)\,dz. \quad \square$$

Residue Thm Let f be analytic on $U\setminus E$, $U \subseteq \mathbb{C}$ open, E discrete subset of U. If γ is a closed path in $U\setminus E$ which is homologous to 0 in U, then

(a) there are only finitely many pts of E at which $\operatorname{Ind}_\gamma$ is nonzero

(b) if these pts are $\{z_1, \dots, z_n\} \subseteq E$, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \operatorname{Ind}_\gamma(z_j)\operatorname{Res}(f, z_j).$$

Pf (a) Choose $r>0$ s.t. $\gamma(I) \subseteq D_r(0)$. Then the bdd cpts of $\mathbb{C}\setminus\gamma(I)$ are inside $D_r(0)$. $\operatorname{Ind}_\gamma$ is nonzero only on (some) bdd cpts of $\mathbb{C}\setminus\gamma(I)$. Thus $\gamma(I) \cup \{\text{cpts of } \mathbb{C}\setminus\gamma(I) \text{ on which } \operatorname{Ind}_\gamma \text{ is nonzero}\}$ is a bdd set K. $K = \mathbb{C} \setminus \{\text{cpts of } \mathbb{C}\setminus\gamma(I) \text{ on which } \operatorname{Ind}_\gamma = 0\}$ so K is closed. Thus K is compact.

Since γ nullhom in U, $K \subseteq U$. Choose for each point of U an open disc containing either no sings, or only one sing at its center. (Possible since E discrete.) This is an open cover of K hence it contains a finite subcover. $\Rightarrow$ only finitely many sings of f in K. ✓(a)

(b) Let $z_1, \dots, z_n$ be the sings of f at which $\mathrm{Ind}_\gamma \neq 0$. For each z_j choose $r_j > 0$ s.t. $D_{r_j}(z_j) \subseteq U - \gamma(I)$. Choose $0 < r < \min\{r_1, \dots, r_n\}$ s.t. $\bigcap_{j=1}^{n} \bar{D}_r(z_j) = \emptyset$. Then set $m_j = \mathrm{Ind}_\gamma(z_j)$ and define 1-cycle $\Gamma = \gamma - \sum_{j=1}^{n} m_j \gamma_j$ with $\gamma_j(t) = z_j + re^{2\pi i t}$ for $t \in [0,1]$.

Have each $\bar{D}_r(z_j) \subseteq K \subseteq U \Rightarrow \mathbb{C} - U \subseteq \bar{D}_r(z_j)$

$\Rightarrow \mathrm{Ind}_{\gamma_j}(z) = 0$ on $z \in \mathbb{C} - U$, and same for γ.

Thus Γ is nullhomologous in U.

Also have $\mathrm{Ind}_\gamma(z_j) = m_j = m_j \,\mathrm{Ind}_{\gamma_j}(z_j)$

and $m_j \,\mathrm{Ind}_{\gamma_j}(z_k) = 0$ for $k \neq j$.

Thus Γ is also nullhomologous in $U - E$ where f is analytic.

By the general Cauchy integral fmla,

$$0 = \int_\Gamma f(z)\,dz = \int_\gamma f(z)\,dz - \sum_{j=1}^{n} \underbrace{m_j}_{\mathrm{Ind}_\gamma(z_j)} \underbrace{\int_{\gamma_j} f(z)\,dz}_{\mathrm{Res}(f, z_j)}$$

$\square$

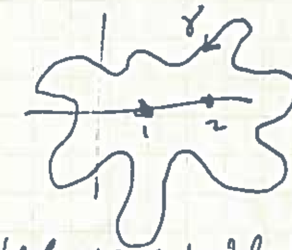
e.g. Let γ be a simple closed path with $1, 2$ inside $\gamma(I)$.

We determine $\displaystyle\int_\gamma \underbrace{\frac{z+1}{(z-1)(z-2)}}_{f(z)}\,dz = 2\pi i\left(\mathrm{Res}(f,1) + \mathrm{Res}(f,2)\right)$.

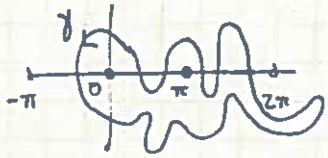
Let $g(z) = \dfrac{z+1}{z-2}$ so that g is analytic at 1 and $f(z) = \dfrac{g(z)}{z-1}$.

Thus $\mathrm{Res}(f,1) = g(1) = -2$. Working similarly with $h(z) = \dfrac{z+1}{z-1}$, get $\mathrm{Res}(f,2) = h(2) = 3$. Thus

$$\int_\gamma \frac{z+1}{(z-1)(z-2)}\,dz = 2\pi i\,(3-2) = 2\pi i$$



e.g.



$f(z) = \frac{1}{\sin z}$ has isolated sings inside γ at 0 and π. The fn $g(z) = \frac{z}{\sin(z)}$ has a removable sing at 0 and the value of the corr analytic fn at 0 is 1. Thus $\operatorname{Res}(f, 0) = 1$. Since $\sin(z) = -\sin(z-\pi)$,

$h(z) = \frac{z-\pi}{\sin(z)} = -\frac{z-\pi}{\sin(z-\pi)}$ has a removable sing at $z=\pi$ and corr value at π is -1. Thus $\operatorname{Res}(f,\pi) = -1$. By the Residue Thm,

$$\int_\gamma \frac{dz}{\sin z} = 0.$$

Counting zeroes and poles

Recall: f meromorphic on U means it is analytic on $U \setminus E$ where $E \subseteq U$ is discrete and f has poles on E.

Thm If f is meromorphic on U and $z_0 \in U$, then

$$\operatorname{Res}(f'/f, z_0) = k,$$

where $k = \begin{cases} \text{order of the zero of } f \text{ at } z_0 \\ -(\text{order of the pole of } f \text{ at } z_0) \\ 0 \quad \text{no zero or pole at } z_0 \end{cases}$.

Pf May factor $f(z) = (z-z_0)^k g(z)$ where g is meromorphic on U with no zero or pole at z_0. Then

$$f'(z) = k(z-z_0)^{k-1} g(z) + (z-z_0)^k g'(z)$$

so

$$\frac{f'(z)}{f(z)} = \frac{k}{z-z_0} + \frac{g'(z)}{g(z)}$$

Since g'/g is analytic at z_0, $\operatorname{Res}(f'/f, z_0) = k$. $\square$

Combined with the residue thm, we get:

Thm Let f be meromorphic on $U \subseteq \mathbb{C}$ open and let γ be a closed path in U homologous to 0 in U. Assume no zeroes or poles of f on $\gamma(I)$, and suppose the zeroes and poles of f at which $\mathrm{Ind}_\gamma \neq 0$ are $z_1, \dots, z_n$. Set $k_j = \begin{cases} \text{order of the zero of } f \text{ at } z_0 \\ -(\text{order of the pole of } f \text{ at } z_0) \end{cases}$ and $m_j = \mathrm{Ind}_\gamma(z_j)$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^n m_j k_j \quad . \quad \square$$

Cor For $U, f, z_1, \dots, z_n, k_1, \dots, k_n, \gamma, m_1, \dots, m_n$'s in Thm, if we create the new path $f \circ \gamma$ then $\sum_{j=1}^n m_j k_j = \mathrm{Ind}_{f\circ\gamma}(0)$.

Pf If $\gamma : [a,b] \to U$ then

$$\mathrm{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z}\,dz = \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t)}{f\circ\gamma(t)}\,dt$$

$$= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\,\gamma'(t)}{f(\gamma(t))}\,dt = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

$$= \sum m_j k_j \quad . \quad \square$$

Cor If γ is a simple closed path, we get the above formulae with $m_j = 1$.

Application Suppose f is meromorphic in a convex open $U \subseteq \mathbb{C}$. Suppose the only zero of f in U is at z_1 of order k, and the only pole of f in U is at z_2, also of order k. Then $\exists$ analytic $g : U - [z_1, z_2] \to \mathbb{C}$ s.t. $f(z) = e^{g(z)}$ for $z \in U - [z_1, z_2]$.
(Call g a logarithm of f.)

Pf Set $V = U \setminus [z_1, z_2]$. If γ is a closed path in V, then γ is homologous to 0 in U since U is convex. Note $\mathrm{Ind}_\gamma(z_1) = \mathrm{Ind}_\gamma(z_2)$ since z_1, z_2 are connected by a line segment in $\mathbb{C} \setminus V$ hence are in the same connected component of $\mathbb{C} \setminus \gamma(I)$. Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \mathrm{Ind}_\gamma(z_1)k - \mathrm{Ind}_\gamma(z_2)k = 0. \quad \circledast$$

Take $z_0 \in V$ fixed and z varying in V. Let γ_z be a path in V beginning at z_0, ending at z. Then

$$h(z) = \int_{\gamma_z} f'/f$$

is independent of the choice of path γ_z by $\circledast$.
Furthermore h is a primitive of f'/f, i.e. $h' = f'/f$, whence

$$\begin{aligned}(fe^{-h})' &= f'e^{-h} - fh'e^{-h} \\ &= f'e^{-h} - f'e^{-h} \\ &= 0.\end{aligned}$$

Thus $fe^{-h} = c$, constant, i.e. $f = ce^h$.
Set $g(z) = h(z) + \log(c)$ to get $e^{g(z)} = e^{h(z)}f(z)e^{-h(z)} = f(z)$ where $\log$ is any branch of the logarithm. $\square$

Homotopy

Facts about approximating closed curves (read pp. 141–144):

For $\gamma_1, \gamma_2 : I \to \mathbb{C}$ curves (cts fns) define

$$\|\gamma_1 - \gamma_2\| = \sup_{t \in I} |\gamma_1(t) - \gamma_2(t)|.$$

Thm 1 If $\gamma : I \to \mathbb{C}$ is a curve, then $\forall \varepsilon > 0$ there is a piecewise linear curve $\tilde{\gamma}$ s.t. $\|\tilde{\gamma} - \gamma\| < \varepsilon$.

Thm 2 If $\gamma : I \to \mathbb{C}$ closed curve, $z \in \mathbb{C} - \gamma(I)$, then $\exists \delta > 0$ s.t. if γ_1, γ_2 are paths with $\|\gamma - \gamma_j\| < \delta$, $j = 1, 2$, then $\mathrm{Ind}_{\gamma_1}(z) = \mathrm{Ind}_{\gamma_2}(z)$.

Defn If $\gamma : I \to \mathbb{C}$ closed curve, $z \in \mathbb{C} - \gamma(I)$, choose $\delta > 0$ as in Thm 2 and $\tilde{\gamma}$ as in Thm 1 (w/ $\varepsilon = \delta$). Set $\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_{\tilde{\gamma}}(z)$.

Thm $\mathrm{Ind}_\gamma(z)$ is a locally constant function of γ.

$U \subseteq \mathbb{C}$ open, $\gamma_0, \gamma_1 : I \to U$, $I = [0,1]$. closed curves

Defn γ_0, γ_1 are homotopic in U if $\exists$ cts $h : I^2 \to U$ s.t.

(a) $h(0,t) = \gamma_0(t) \quad \forall t \in I$

(b) $h(1,t) = \gamma_1(t) \quad \forall t \in I$

(c) $h(s,0) = h(s,1) \quad \forall s \in I$

Write $\gamma_s(t) := h(s,t)$

Thm $\forall s_0 \in I, \varepsilon > 0\ \exists \delta > 0$ s.t. $\|\gamma_s - \gamma_{s_0}\| < \varepsilon$ for $|s - s_0| < \delta$.

Pf Cts fn on cpt set is unif cts. $\square$

<u>Thm</u> If $\gamma_0, \gamma_1 : I \to U$ are homotopic in U then
$$\operatorname{Ind}_{\gamma_0}(z) = \operatorname{Ind}_{\gamma_1}(z) \quad \forall z \in \mathbb{C} \setminus U.$$

<u>Pf</u> By previous thms, $\forall s_0 \in I \ \exists \varepsilon > 0$ s.t. $\|\gamma_s - \gamma_{s_0}\| < \varepsilon \Rightarrow \operatorname{Ind}_{\gamma_s}(z) = \operatorname{Ind}_{\gamma_{s_0}}(z)$ and $\exists \delta > 0$ s.t. $|s - s_0| < \delta \Rightarrow \|\gamma_s - \gamma_{s_0}\| < \varepsilon$. As such, $\operatorname{Ind}_{\gamma_s}(z)$ is constant on $(s_0 - \delta, s_0 + \delta)$ $\forall s \in I$. Thus the set on which $\operatorname{Ind}_{\gamma_s}(z)$ takes any given value is open. Since I is connected, $\operatorname{Ind}_{\gamma_s}(z)$ is constant. $\square$

<u>Thm</u> If $\gamma_0, \gamma_1 : I \to U$ are homotopic closed paths in U, then $\Gamma = \gamma_1 - \gamma_0$ is nullhomologous in U and $\int_{\gamma_1} f = \int_{\gamma_0} f$. $\square$

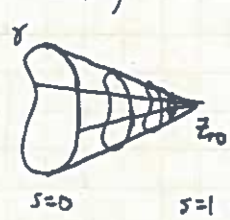
<u>e.g.</u>

are $\underbrace{\text{htpic in } \mathbb{C} - \{0\}}_{\text{explicit htpy}}$ but $\underbrace{\text{not in } \mathbb{C} - \{3/2\}}_{\text{different indices at } 3/2}$.

<u>Defn</u> A connected open set U is <u>simply connected</u> if every closed curve in U is homotopic to a point (i.e. to a constant curve) in U.

<u>Prop</u> Convex open sets are simply connected.

<u>Pf</u> $U \subseteq \mathbb{C}$ convex open, $z_0 \in U$. If $\gamma : I \to U$ closed curve in U, define $h(s,t) = (1-s)\gamma(t) + s z_0$, the "straight line homotopy."



$\square$

Thm $U \subseteq \mathbb{C}$ open and connected then TFAE:

(a) U is simply connected
(b) every cycle in U is nullhomologous
(c) $\mathbb{C} \setminus U$ has no bounded components
(d) $\forall$ cycles Γ in U and analytic fns f on U, $\int_\Gamma f = 0$
(e) every analytic fn f on U has an antiderivative
(f) every harmonic fn f on U has a harmonic conjugate
(g) every nonvanishing analytic fn f on U has an analytic logarithm
(h) every nonvanishing analytic fn f on U has an analytic square root.

Pf (a) $\Rightarrow$ (b): $\mathrm{Ind}_{*}(z) = 0$ on $\mathbb{C} \setminus * \supseteq \mathbb{C} \setminus U$.

(b) $\Rightarrow$ (c): ~~If $\mathbb{C} \setminus U$ has a bdd cpt C, then C is contained in a closed bdd $A \subseteq \mathbb{C} \setminus U$ s.t. $B = (\mathbb{C} \setminus U) \setminus A$ is also closed. (Check!)~~

Idea Build a curve in U around a bdd cpt of $\mathbb{C} \setminus U$ with index 1 in the cpt.

(c) $\Rightarrow$ (d): $z \in \mathbb{C} \setminus U$ is in the unbdd cpt of $\mathbb{C} \setminus \Gamma(I)$ so $\mathrm{Ind}_\Gamma(z) = 0$ ~~analog's integral~~, so Cauchy's Thm gives $\int_\Gamma f = 0$.

(d) $\Rightarrow$ (e): Fix $z_0 \in U$. $g(z) = \int_{\gamma_z} f(w)\,dw$ for γ_z any path z_0 to z is an antideriv.

(e) $\Rightarrow$ (f): Read pf of Thm 3.5.7.

(f) $\Rightarrow$ (g): For f analytic nonvan on U, $\log|f|$ is harmonic. Let g be analytic with $\mathrm{Re}(g) = \log|f|$. Then $|e^g| = |f|$ on U $\Rightarrow$ $|fe^{-g}| = 1$ on U hence fe^{-g} constant by Max Modulus Thm say with $fe^{-g} = a \in \mathbb{C} - \{0\}$. Choose a log b of a, $a = e^b$. Then $f = e^{g+b}$ so $g+b$ is an analytic log.

(g) $\Rightarrow$ (h): If f has analytic log h, then $e^{h/2}$ is an analytic square root of f

(h) $\Rightarrow$ (a): Wait for Riemann Mapping Thm! $\square$

Htpy for non-closed curves:

Ex:

Thm If γ_0, γ_1 are homotopic paths in U, connecting z_0, z_1, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for all f analytic on U.

Thm U conn'd open, $z_0, z_1 \in U$. U is simply conn'd iff any two curves connecting z_0, z_1 are htpic in U.

Calculus of Residues

Thm If $f(z) = \dfrac{g(z)}{(z-z_0)^k}$ where g is analytic in a nbhd of z_0, then $\mathrm{Res}(f, z_0) = \dfrac{g^{(k-1)}(z_0)}{(k-1)!}$ is the coeff of $(z-z_0)^{k-1}$ in the power series for g at z_0.

Pf Clear. $\square$

e.g. If $f(z) = \dfrac{g(z)}{z-z_0}$ then $\mathrm{Res}(f, z_0) = g(z_0)$

If $f(z) = \dfrac{g(z)}{(z-z_0)^2}$ then $\mathrm{Res}(f, z_0) = g'(z_0)$.

e.g. $f(z) = \dfrac{z^2+1}{(z-1)(z^2-2z+5)} = \dfrac{g(z)}{z-1}$ for $g(z) = \dfrac{z^2+1}{z^2-2z+5}$ analytic at $z=1$. Thus $\mathrm{Res}(f, 1) = g(1) = \frac{1}{2}$.

e.g. $f(z) = \dfrac{\sin(z)}{z^2}$ then $\mathrm{Res}(f, 0) = \sin'(0) = \cos(0) = 1$.

Residue of a Quotient

Thm Suppose $f(z) = \dfrac{p(z)}{h(z)}$ for p, h analytic at z_0, h has a zero of order k at z_0. If we write $h(z) = (z-z_0)^k q(z)$ for q analytic at z_0 w/ $q(z_0) \neq 0$, then $\mathrm{Res}(f, z_0) = c_{k-1}$ for c_{k-1} the coeff of $(z-z_0)^{k-1}$ in the power series exp'n of $g(z) = \dfrac{p(z)}{q(z)}$.

Note May compute c_{k-1} by long division of power series.

Suppose $k=1$. Then $\mathrm{Res}(f, z_0) = c_0 = \dfrac{p(z_0)}{q(z_0)}$

Since $h(z) = (z-z_0)q(z)$, $q(z_0) = h'(z_0)$ and we get

Cor For p, h analytic at z_0 where h has a zero of order 1 at z_0,
$$\mathrm{Res}(p/h, z_0) = p(z_0)/h'(z_0). \quad \square$$

e.g. $\operatorname{Res}(1/\sin z, 0) = \dfrac{1}{\cos(0)} = 1.$

e.g. $f(z) = \dfrac{1}{e^z - 1 - z} = \dfrac{1}{z^2 q(z)}$ for $q(z) = \frac{1}{2} + \frac{z}{3!} + \cdots$.

To find $\operatorname{Res}(f,0)$, seek linear coeff of $1/q$:

$$\begin{array}{r l}
 & 2 - \frac{4}{6}z + \cdots \\ \hline
\left(\frac{1}{2} + \frac{z}{3!} + \cdots\right) \Big) & 1 \\
 & 1 + \frac{2z}{3!} + \cdots \\ \hline
 & -\frac{2z}{3!} + \cdots \\
 & +\frac{4z}{6} + \cdots \\ \hline
 & ?z^2 + \cdots
\end{array}$$

$\Longrightarrow \operatorname{Res}(f,0) = \dfrac{-2}{3}$.

e.g. $\operatorname{Res}\left(\dfrac{\cot z}{z^2}, 0\right) = ?$

$\dfrac{\cot z}{z^2}$ has a pole of order 3 at 0. Set $p(z) = \cos z$, $h(z) = z^2 \sin z$,

$$q(z) = \frac{h(z)}{z^3} = \frac{\sin z}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots$$

To compute c_2 for p/q:

$$\begin{array}{r l}
 & 1 - \frac{z^2}{3} + \cdots \\ \hline
1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots \Big) & 1 - \frac{z^2}{2} + \frac{z^4}{12} - \cdots \\
 & 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots \\ \hline
 & -\frac{z^2}{3} + \frac{9z^4}{120} - \cdots \\
 & -\frac{z^2}{3} + \frac{z^4}{18} + \cdots \\ \hline
 & ?z^4
\end{array}$$

$\Longrightarrow \operatorname{Res}\left(\dfrac{\cot z}{z^2}, 0\right) = -\dfrac{1}{3}$.

Evaluating integrals using residues

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

so $\cos\theta = \left[\frac{z+z^{-1}}{2}\right](e^{i\theta})$, $\sin\theta = \left[\frac{z-z^{-1}}{2i}\right](e^{i\theta})$.

Thus if f is a rational expression in $\cos\theta$, $\sin\theta$ we may evaluate $\int_0^{2\pi} f(\theta)\,d\theta$ by letting $g(z)$ be the rat'l fn in z where $\cos\theta$ replaced by $\frac{z+z^{-1}}{2}$, $\sin\theta$ by $\frac{z-z^{-1}}{2i}$ so that $g(e^{i\theta}) = f(\theta)$ and

$$\int_\gamma \frac{g(z)}{iz}\,dz \quad \text{for } \gamma(\theta) = e^{i\theta} \text{ on } [0,2\pi].$$
$$= \int_0^{2\pi} f(\theta)\,d\theta.$$

e.g. $f(\theta) = \frac{1}{2+\sin\theta} \rightsquigarrow \frac{1}{2+(z-z^{-1})/2i} = \frac{2iz}{4iz+z^2-1} =: g(z)$

Dividing by iz, want to compute $\int_{|z|=1} \frac{2}{z^2+4iz-1}\,dz$

$= \int_{|z|=1} \frac{2}{(z+(2-\sqrt{3})i)(z+(2+\sqrt{3})i)}\,dz$ Only pole in $D_1(0)$ is $(-2+\sqrt{3})i$. The residue here is $\frac{2}{(-2+\sqrt{3})i+(2+\sqrt{3})i} = \frac{-\sqrt{3}}{3}i$

By the residue thm,

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = 2\pi i \cdot \left(\frac{-\sqrt{3}}{3}i\right)\cdot 1$$
$$= \frac{2\sqrt{3}}{3}\pi.$$

Improper Integrals

Call $\lim\limits_{x\to\infty}\int_{-x}^{x} f(t)\,dt$ the principal value of $\int_{-\infty}^{\infty} f(t)\,dt$, denoted P.V.$\int_{-\infty}^{\infty} f(t)\,dt$. If $\int_{\mathbb{R}} f$ exists $\left(= \lim\limits_{x,y\to\infty}\int_{-x}^{y} f(t)\,dt\right)$ then

$$\text{PV}\int_{\mathbb{R}} f = \int_{\mathbb{R}} f.$$

Suppose $U \subseteq \mathbb{C}$ open, $H \subseteq U$ for $H = \{z \mid \operatorname{Im} z \geq 0\}$ or $\{z \mid \operatorname{Im} z \leq 0\}$.
f meromorphic on U with no sings on $\mathbb{R}$.
Suppose $\exists\, p > 1$, $R, C > 0$ s.t. $|f(z)| \underset{⊛}{\leq} C|z|^{-p}$ for $z \in H$, $|z| > R$.

Let γ_r be

or

Then $\int_{\gamma_r} f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_0^{\pi} f(re^{it})\,ire^{it}\,dt$ (in upper case

If $r > R$, then ⊛ gives $|f(re^{it})\,ire^{it}| \leq Cr^{1-p}$ and so

$$\left|\int_0^{\pi} f(re^{it})\,ire^{it}\,dt\right| \leq \pi C r^{1-p} \xrightarrow[r\to\infty]{} 0 \quad \text{since } p > 1.$$

Thus $\lim\limits_{r\to\infty}\int_{\gamma_r} f(z)\,dz = \text{P.V.}\int_{-\infty}^{\infty} f(t)\,dt$ (in fact, $\int_{\mathbb{R}} f$ converges under ⊛).

By the residue theorem, $\int_{\gamma_r} f(z)\,dz = 2\pi i \sum\limits_{\substack{z \text{ inside} \\ \gamma_r,\ \text{sing of } f}} \operatorname{Res}(f,z)$

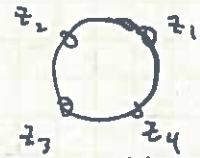
By ⊛, no sings outside $|z| = R$, so only finitely many sings in upper half plane $\Longrightarrow$ integral ind of r for $r > R$ and $= 2\pi i \sum\limits_{\substack{z\in H \\ \text{sing}}} \operatorname{Res}(f,z)$ (for lower H, $-2\pi i \sum \operatorname{Res}$)

<u>e.g.</u> $\displaystyle\int_{\mathbb{R}} \frac{x^2}{1+x^4}\,dx = ?$

Set $f(z) = \dfrac{z^2}{1+z^4} = \dfrac{z^{-2}}{z^{-4}+1}$. Set $R>1$. Get

$|f(z)| \le \dfrac{|z|^{-2}}{1-R^{-4}}$ for $|z| \geq R$ so (*) satisfied w/ $p=2$, $C = \dfrac{1}{1-R^{-4}}$

The poles of f are at 4th roots of -1:



$f(z) = \dfrac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$ ⤷ eval at z_j w/ $z-z_j$ removed to get $\mathrm{Res}(f, z_j)$ $j=1,2$

$\mathrm{Res}(f, z_1) = -\dfrac{\sqrt{2}}{8}(1+i)$

$\mathrm{Res}(f, z_2) = \dfrac{\sqrt{2}}{8}(1-i)$

$\Sigma\,\mathrm{Res} = -\dfrac{\sqrt{2}}{4}\,i \implies \displaystyle\int_{\mathbb{R}} \frac{x^2}{1+x^4}\,dx = \frac{\sqrt{2}}{2}\pi.$

<u>Read</u> e.g. 5.2.5, 5.2.6.

Summing Infinite Series

Hope: Compute $\sum_{n=-\infty}^{\infty} f(n)$, f analytic with isolated sings on $\mathbb{C}$

Idea: · Find g with simple pole with residue 1 at each $n \in \mathbb{Z}$.

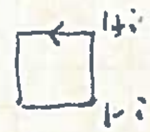
· Then fg has a simple pole with residue $f(n)$ at $n \in \mathbb{Z}$.

· Find $\{\gamma_N\}_N$ simple closed paths s.t. sings of fg contained in γ_N for $N \gg 0$ and s.t. $\int_{\gamma_N} fg \longrightarrow 0$ as $N \to \infty$.

· Then, by Residue Thm, $\sum \operatorname{Res}(fg, z_i) = 0$

If f has only fin many sings, none of which are integers, then $\sum f(n) = -\sum_{\substack{w_i \text{ sing} \\ \text{of } f}} \operatorname{Res}(fg, w_i)$.

Good choice of g is $g(z) = \pi \cot(\pi z)$, which has a simple pole with residue 1 at each integer, and no other poles.

For $N \in \mathbb{Z}^+$, take γ_N:  $(N+\frac{1}{2})\cdot$ 

These capture all sings and g is bdd on γ_N for $N \gg 0$:

<u>Lemma</u> $\exists R > 0$ s.t. $|\cot(\pi z)| \le 2$ on $\gamma_N(I)$ for $N \ge R$.

<u>Pf</u> $$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i\,\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}}$$

$$= i\,\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = i\,\frac{e^{2\pi i x}e^{-2\pi y}+1}{e^{2\pi i x}e^{-2\pi y}-1} \quad \text{for } z = x+iy.$$

Thus $|\cot(\pi z)| = \left|\dfrac{e^{2\pi i x}e^{-2\pi y}+1}{e^{2\pi i x}e^{-2\pi y}-1}\right|$. If $x = N+\frac{1}{2}$, $e^{2\pi i x} = -1$

$\Rightarrow |\cot(\pi z)| = \left|\dfrac{1-e^{-2\pi y}}{1+e^{-2\pi y}}\right| \le 1$ on vertical sides.

On horizontals, max for $e^{2\pi i x} = 1$ i.e. $x \in \mathbb{Z}$.

Thus $\max \left| \dfrac{e^{\pm(2N+1)\pi}+1}{e^{\pm(2N+1)\pi}-1} \right| \xrightarrow[N\to\infty]{} 1 \Rightarrow \exists N$ s.t. <2 for $N > M$. $\square$

<u>Lemma</u> With γ_N as above, $\displaystyle\int_{\gamma_N} \frac{\pi \cot \pi z}{z} = 0$ for each $N \in \mathbb{Z}^+$

<u>Pf</u> $\displaystyle\int_{\gamma_N} \underbrace{\frac{\pi \cot(\pi z)}{z}}_{h(z)} dz = 2\pi i \sum_{-N}^{N} \operatorname{Res}(h, n)$

$\operatorname{Res}(h, 0) = 0$ since h even (HW)

$\underset{\substack{\neq \\ 0}}{\operatorname{Res}(h, n)} = \dfrac{\pi \cos(\pi n)/n}{\pi \cos(\pi n)} = \dfrac{1}{n}$ odd $\Rightarrow \displaystyle\int_{\gamma_N} = 0$. $\square$

~ simple pole

<u>Thm</u> Suppose f is analytic on $\mathbb{C}$ except at $E = \{z_1, \dots, z_m\}$ iso sings. Suppose $\exists R, M > 0$ s.t. $|f(z)| \underset{(*)}{\leq} \dfrac{M}{|z|}$ for $|z| \geq R$.

Then $\displaystyle\lim_{N\to\infty} \sum_{n \in [-N,N] \cap \mathbb{Z} \setminus E} f(n) = -\sum_{j=1}^{m} \operatorname{Res}(\pi f(z) \cot(\pi z), z_j)$.

<u>Pf</u> The sings of $\pi f(z) \cot(\pi z)$ are at $\mathbb{Z} \cup E$. For $n \in \mathbb{Z} \setminus E$, $\operatorname{Res} = f(n)$ since $\pi \cot(\pi z)$ simple pole w/ res 1 at n, f analytic at n. For $N \gg 0$ s.t. E inside γ_N,

$$\underbrace{\frac{1}{2\pi i} \int_{\gamma_N} \pi f(z) \cot(\pi z)\, dz}_{} = \sum_{n \in [-N,N] \cap \mathbb{Z} \setminus E} f(n) + \sum_{j=1}^{m} \operatorname{Res}(\pi f(z) \cot(\pi z), z_j).$$

Thus suffices to show $\to 0$ as $N \to \infty$.

Choose $R, M > 0$ s.t. $(*)$ holds, $|\cot(\pi z)| < 2$ on γ_N for $N \geq R$, and $|z_j| < R$, $j = 1, \dots, m$. Then f is analytic in

$$A = \{z \mid R < |z| < \infty\}$$

and f vanishes at ∞.

Thus f has a Laurent exp'n of the form

$$f(z) = \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots \quad \text{on } A.$$

Since $\int_{\gamma_N} \frac{\pi \cot(\pi z)}{z} = 0$, get

$$\frac{1}{2\pi i}\int_{\gamma_N} \pi f(z)\cot(\pi z)\,dz = \frac{1}{2\pi i}\int_{\gamma_N} \pi\left(f(z) - \frac{c_{-1}}{z}\right)\cot(\pi z)\,dz$$

get smart

Now $f(z) - \frac{c_{-1}}{z} = \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} + \cdots = \frac{q(z)}{z^2}$ in A where q is analytic in A with $q(z) \longrightarrow c_{-2}$ as $z \to \infty$.

Thus for some $R_1 > R$, $|q(z)| < M_1$ on $\mathbb{C} \setminus D_{R_1}(0)$.

and integrand modulus bdd by $2MM_1/|z|^2$.

Since $|z| > N$ on γ_N, length bd gives

$$\left|\frac{1}{2\pi i}\int_{\gamma_N} \pi f(z)\cot(\pi z)\,dz\right| \leq \frac{2MM_1\,(8N+4)}{N^2} \xrightarrow[N\to\infty]{} 0. \quad \square$$

<u>Thm</u> $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

<u>Pf</u> Preceding thm with $f(z) = \frac{1}{z^2}$ gives

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = -\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2}, 0\right)$$

Have seen $\operatorname{Res}\left(\frac{\cot(z)}{z^2}, 0\right) = -\frac{1}{3} \implies = \frac{\pi^2}{3}$.

Thus $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\frac{\pi^2}{3} = \frac{\pi^2}{6}$. $\square$

<u>Thm</u> $\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$ $\left(\frac{\pi^2}{6}, \frac{\pi^4}{90}, \frac{\pi^6}{945}, \ldots\right)$

Conformal Mappings

Conformal = angle preserving

If $h = u + iv : U \to \mathbb{C}$, $\quad Jh := \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

If $\gamma = x + iy : [a,b] \to \mathbb{C}$ with $\gamma(t_0) = z_0$, then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is the tangent vector to γ at z_0

By chain rule, $(h \circ \gamma)'(t_0) = Jh(z_0) \cdot \gamma'(t_0)$

i.e. $(h \circ \gamma)' = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

Defn h is <u>conformal</u> at z_0 if angle b/w $Jh(z_0) \cdot a$ & $Jh(z_0) \cdot b$ $=$ angle(a, b) $\forall a, b \in \mathbb{R}^2 \setminus 0$, i.e.

$$\langle a, b \rangle = \langle Jh(z_0) \cdot a, Jh(z_0) \cdot b \rangle$$

If $h : U \longrightarrow V$ conformal at all $z_0 \in U$ and surj, call h <u>conformal</u>.

Thm h is conformal at z_0 iff it has a nonzero cpx deriv at z_0.

Pf If h has a cpx deriv at z_0 then

$$Jh(z_0) = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$=$ dilation $\cdot$ rotation so conformal

Converse: reading/exc. $\square$

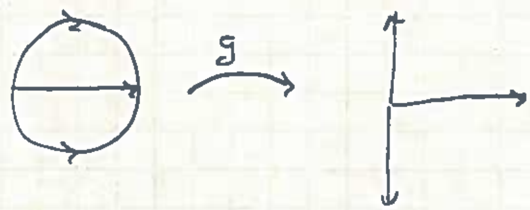
<u>Thm</u> If h is an injective conformal map $U \to V$ then h has an inverse $h^{-1}: V \to U$ also conformal.

<u>Pf</u> Inverse mapping theorem. $\square$

e.g.

$$\begin{array}{ccc} z & \longmapsto & \frac{1+z}{1-z} \\ g: \mathbb{C}\setminus\{1\} & \longleftrightarrow & \mathbb{C}\setminus\{-1\} : g^{-1} \\ -\frac{1-w}{1+w} & \longleftarrow\!\shortmid & w \end{array}$$

When $|z|<1$, and $\mathrm{Re}(z)>0$ (and vice versa) so g restricts to a conformal equiv of $D_1(0) \longrightarrow$ right half plane



Thus $h: D_1(0) \longrightarrow H$ conf equiv

$$z \longmapsto i\,\frac{1+z}{1-z}$$

Now $H \longrightarrow$ 1st quadrant conv equiv

$$re^{i\theta} = z \longmapsto \sqrt{z} = \sqrt{r}\, e^{i\theta/2} \qquad -\pi < \theta < \pi$$

so $f: D_1(0) \longrightarrow$ 1st quad conf equiv

$$z \longmapsto \sqrt{i\,\frac{1+z}{1-z}}$$

e.g. $\mathrm{Log}: \{\mathrm{Re}\, z > 0\} \longrightarrow \{-\pi/2 < \mathrm{Im}(z) < \pi/2\}$ conf equiv

$$\begin{tikzcd} \{\mathrm{Re}\, z>0\} \arrow[r] & \{-\pi/2 < \mathrm{Im}(z) < \pi/2\} \arrow[d, "/\pi/2"] \\ D_1(0) \arrow[u, "g"] \arrow[r] & \{-1 < \mathrm{Im}(z) < 1\} \end{tikzcd}$$

$$z \longmapsto \frac{2}{\pi}\,\mathrm{Log}\left(\frac{1+z}{1-z}\right)$$

The Riemann Sphere

Idea $\mathbb{C} \cup \{\infty\}$ is a sphere

Coordinatize: write $S^2 := \mathbb{C} \cup \{\infty\}$.

$$\begin{tikzcd} \mathbb{C} \setminus 0 \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C} & z \arrow[d, mapsto] \\ \mathbb{C} \arrow[r] & S^2 & z \end{tikzcd}$$

$$z \longmapsto \begin{cases} \frac{1}{z} = w & z \neq 0 \\ \infty & z = 0 \end{cases}$$

An open disc in S^2 centered at ∞ is $\{w \mid |w| < r\} = \{z \mid |z| > 1/r\}$

An open set in S^2 is a set U s.t. every pt in U is the center of some open disc contained in U.

A nbhd of ∞ = open set containing ∞

= $\{\infty\} \cup$ open subset of $\mathbb{C}$ containing the complement of a closed bdd disc centered at 0.

If $f: U \longrightarrow S^2$ for $U \subseteq S^2$ open.

$\lim_{z \to z_0} f(z) = L$ means $\forall$ nbhd W of L, there is a deleted nbhd V of z_0 s.t. $V \subseteq U$ and $f(V) \subseteq W$.

__Analytic functions on S^2__ "analytic at z_0" = defined and analytic in a nbhd of z_0.

__Defn__ Say $f(z)$ is __analytic at ∞__ if $f(1/w)$ is analytic at $w=0$ (i.e. defined and analytic in a deleted nbhd of 0 with removable sing at 0).

__e.g.__ $f(z) = \frac{1-z}{1+z}$ is analytic at ∞ if we set $f(\infty) = -1$.

Indeed, $f(1/w) = \frac{1 - 1/w}{1 + 1/w} = \frac{w-1}{w+1}$ which is analytic at $w=0$.

Analytic functions $U \to S^2$.

For $U \subseteq S^2$ open, $f: U \to S^2$, call f analytic at z_0 with $f(z_0) = \infty$ iff $1/f$ is analytic at z_0.

Write $s: S^2 \longrightarrow S^2$

$$z \longmapsto \begin{cases} 1/z & z \neq 0, \infty \\ \infty & z = 0 \\ 0 & z = \infty \end{cases}$$

so f is analytic at ∞ iff $f \circ s$ analytic at 0

f is analytic at z_0, $f(z_0) = \infty$ iff $s \circ f$ analytic at z_0.

<u>Thm</u> s is a conformal equivalence $S^2 \to S^2$. $\square$

Meromorphic fn f on $\mathbb{C}$ is said to have a pole of order n at ∞ if $f(1/w)$ has a pole of order n at 0. (Allow $n = 0$.)

<u>Thm</u> Each meromorphic fn on $\mathbb{C}$ with a pole at ∞ defines an analytic fn $\tilde{f}: S^2 \to S^2$, and each analytic function $S^2 \to S^2$ arises in this way.

<u>Pf</u> Suppose f meromorphic on $\mathbb{C}$ with a pole at ∞.

Define $\tilde{f}(z) = \begin{cases} f(z) & z \text{ not a pole} \\ \infty & z \text{ a pole of pos order} \end{cases}$

(If f has a pole of order 0 at ∞, then $f(1/w)$ has rem sing at 0 and $\tilde{f}(\infty)$ is the value at $w=0$ making $f(1/w)$ analytic.)

Check $\tilde{f}$ analytic everywhere.

Suppose $g: S^2 \to S^2$ analytic. Where $g(z_0) \neq \infty$, g is analytic as a cpx valued fn. At $z_0 \in \mathbb{C}$ where $g(z_0) = \infty$, $\frac{1}{g(z)}$ is analytic hence $g(z)$ has a pole. $\square$

Complex Projective Space

$$\mathbb{CP}^1 = \left(\mathbb{C}^2 \setminus \{(0,0)\}\right) \Big/ (z,w) \sim (\lambda z, \lambda w) \text{ for } \lambda \in \mathbb{C} \setminus \{0\}$$

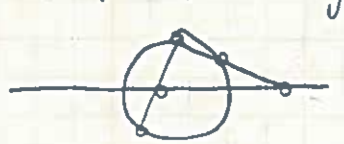
$$= \left\{ [z:w] \;\middle|\; (z,w) \in \mathbb{C}^2 \setminus \{(0,0)\} \right\} \text{ where } [z:w] = \{\lambda(z,w) \mid \lambda \in$$

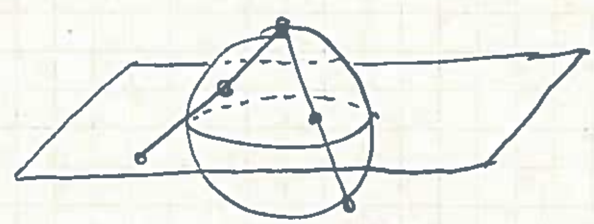
$$\phi : \mathbb{CP}^1 \setminus \{[1:0]\} \longrightarrow \mathbb{C} \longleftarrow \mathbb{CP}^1 \setminus \{[0:1]\} : \psi$$

$$[z:w] \longmapsto \frac{z}{w} \qquad \frac{w}{z} \longmapsfrom [z:w]$$

Again represents $\mathbb{CP}^1$ as two copies of $\mathbb{C}$ glued along $\mathbb{C} \setminus 0$.
so $\mathbb{CP}^1 \cong S^2$.

Stereographic Projection





Linear Fractional Transformations

$$h(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

(If $ad-bc=0$, then $h(z) = \frac{b}{d}$ $\forall z$)

Regard $h: S^2 \longrightarrow S^2$

$$\infty \longmapsto \frac{a}{c}$$
$$-\frac{d}{c} \longmapsto \infty$$

Want to understand $Aut(U)$, conformal automorphisms of $U \subseteq S^2$, esp. $U = S^2$.

Thm Every lin frac trans is a conf aut of S^2. Conversely, each conf aut of S^2 is either an affine trans $L(z) = az+b$ or a comp'n $L_1 \circ s \circ L_2$, L_1, L_2 affine, $s(z) = \frac{1}{z}$. Hence each conf aut of S^2 is a lin frac trans.

pf If $h(z) = \frac{az+b}{cz+d} = w$, then $z = \frac{-dw+b}{cw-a} =: g(w)$.

$$h \circ g(w) = \frac{(ad-bc)w}{ad-bc} = w \text{ since } ad-bc \neq 0.$$

(Also check $w = \infty, -a/c$ and $g \circ h = id$.)

For converse, first suppose $f(\infty) = \infty$. Then $q = s \circ f \circ s : z \longmapsto \frac{1}{f(1/z)}$ is conf and $q(0) = 0$. This must be a 0 of order 1 ($q' \neq 0$) so $f(1/z)$ has a pole of order 1 at 0 and is analytic and finite at every other pt of S^2 $\Longrightarrow$ f has a pole of order 1 at ∞ and is analytic and finite on $\mathbb{C}$. Thus $\frac{f(z)-f(0)}{z}$ is analytic and finite on S^2 $\Longrightarrow$ const on S^2 (Liouville). For const a, $f(z) = az + f(0)$, affine.

Suppose $f(\infty) = k \neq \infty$. Set $p(z) = \frac{1}{z-k}$. Then

$$p \circ f(z) = \frac{1}{f(z)-k} \quad \text{conf aut of } S^2 : \infty \mapsto \infty.$$

Thus $p \circ f(z) = az + b \Rightarrow f(z) = \frac{1}{az+b} + k = \frac{akz + bk + 1}{az+b}$. $\square$

<u>Lines and Circles</u>

Circle in S^2 = circle in $\mathbb{C}$ or line in $\mathbb{C} \cup \{\infty\}$.

<u>Thm</u> Each lin frac trans takes circles in S^2 to circles in S^2.

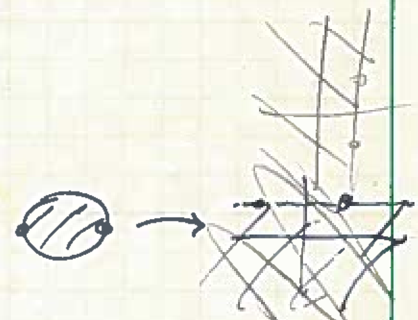
<u>Pf</u> Suffices to check affine trans and s. ✓

Lines/circles are solns to $a|z|^2 + \bar{w}z + w\bar{z} + b = 0$ for some $a, b \in \mathbb{R}$, $w \in \mathbb{C}$ (HW). Applying s to such a soln gives z satisfying $a|z|^{-2} + \bar{w}z^{-1} + w\bar{z}^{-1} + d = 0$

$$\underset{\cdot z\bar{z}}{\Rightarrow} a + \bar{w}z + w\bar{z} + d|z|^2 = 0 \quad ✓$$

<u>e.g.</u> $h(z) = \frac{2z}{z-i}$

$h(1) = 1+i$, $h(-1) = 1-i$, $h(i) = \infty$.



$h(\partial D, (0)) = \{z \mid \mathrm{Re}(z) = 1\} \cup \{\infty\}$

$h(0) = 0 \Rightarrow h(D, (0)) = \{z \mid \mathrm{Re}(z) < 1\}$.

<u>3-Point Thm</u> Given two ordered triples $(w_1, w_2, w_3), (z_1, z_2, z_3) \in (S^2)^{\times 3}$ $\exists!$ LFT h s.t. $h(w_j) = z_j$, $j = 1, 2, 3$.

<u>Pf</u> First take $(w_1, w_2, w_3) \in \mathbb{C}^3 \longrightarrow (0, 1, \infty)$ via $h(z) = \frac{(w_2 - w_3)(z - w_1)}{(w_2 - w_1)(z - w_3)}$

If $w_3 = \infty$, take $h(z) = \frac{z - w_1}{w_2 - w_1}$ to still map to $(0, 1, \infty)$.

Then if $(w_1, w_2, w_3) \xrightarrow{h_1} (0, 1, \infty) \xleftarrow{h_2} (z_1, z_2, z_3)$ set $h_2^{-1} \circ h_1$

as desired. Now suppose g is another such trans.

Then $f = h_2 \circ g \circ h_1^{-1} : (0,1,\infty) \to (0,1,\infty)$.

$$\overset{\parallel}{\frac{az+b}{cz+d}} \qquad \begin{aligned} f(0) = 0 &\Rightarrow b = 0 \\ f(\infty) = \infty &\Rightarrow c = 0 \\ f(1) = 1 &\Rightarrow \frac{a}{d} = 1 \Rightarrow f = id. \end{aligned}$$

$$\Rightarrow g = h_2^{-1} \circ f \circ h_1 = h_2^{-1} \circ h_1 = h \qquad \square$$

<u>Thm</u> For $D = D_1(0)$, $w \in D$, the LFT $h_w(z) = \frac{z-w}{1-\bar{w}z}$ satisfies $h_w(0) = -w$, $h_w(w) = 0$, $h_w(D) = D$, fixes ∂D pointwise. The conformal automorphisms of D are the LFTs of the form $h(z) = u h_w(z)$ for some $|u| = 1$, $|w| < 1$.

Pf p. 201. $\square$

Automorphisms

An LFT is an aut. of $\mathbb{C}$ when $\infty \mapsto \infty$ i.e. $c=0$ so

$$f(z) = az+b \text{ is affine.}$$

In fact, these are all bianalytic fns $\mathbb{C} \to \mathbb{C}$. (HW)

If $f(z) = az+b$, $g(z) = a'z+b'$, then

$$(f \circ g)(z) = aa'z + (ab'+b)$$

$$f^{-1}(z) = a^{-1}z - a^{-1}b \quad \rightsquigarrow \text{ group law.}$$

Easier: $f(z) = az+b \longleftrightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

whence matrix mult'n, inversion give the ops.

Defin $P := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a,b \in \mathbb{C},\ a \neq 0 \right\} \cong \mathrm{Aut}(\mathbb{C})$

(parabolic group

Defin Levi component $M = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{C},\ a \neq 0 \right\}$ = (complex) dilations

unipotent radical $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{C} \right\}$ = translations

Prop $P = MN = NM$ and M normalizes N: $m^{-1}nm \in N$ $\forall m \in M$, $n \in N$.

Pf Check eq'ns. $\square$

Moral
- affine = dil'n $\circ$ trans = trans $\circ$ dilation
- dilation-then-trans-then-inv-dil'n is a trans.

$$\mathrm{Aut}(S^2) \cong \{\text{LFT's}\} \xleftarrow{\text{hom}} GL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{l} a,b,c,d \in \mathbb{C} \\ ad-bc \neq 0 \end{array} \right\}$$

Kernel $= \mathbb{C}^\times I = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}^\times \right\}$

So 1st iso thm $\Rightarrow$ $\mathrm{Aut}(S^2) \cong GL_2(\mathbb{C}) / \mathbb{C}^\times I =: PGL_2(\mathbb{C})$.

Define $SL_2(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}, ad-bc = 1 \right\}$

Let $G = GL_2(\mathbb{C})$, $K = \mathbb{C}^\times I$, $H = SL_2(\mathbb{C})$. Note $H \cap K = \{\pm I\}$

Now $G = HK$ b/c $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in H} \cdot \underbrace{\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}}_{\in K}$

Thus $G/K = HK/K \cong H/H\cap K = SL_2(\mathbb{C}) / \{\pm I\} =: PSL_2(\mathbb{C})$.

Hence $\boxed{\mathrm{Aut}(S^2) \cong PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})}$.

The rotation subgp

$\mathrm{Rot}(\text{unit sphere in } \mathbb{R}^3) \cong SO_3(\mathbb{R}) = \{ m \in M_{3\times 3}(\mathbb{R}) \mid \det m = 1, \; m^T m = I \}$

$\cong \mathrm{Rot}(S^2)$

$SU_2(\mathbb{C}) := \{ m \in M_{2\times 2}(\mathbb{C}) \mid \bar{m}^T m = I, \; \det m = 1 \}$

$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a,b \in \mathbb{C}, \; |a|^2 + |b|^2 = 1 \right\} \leq SL_2\mathbb{C}$

$= S^3$ with group structure!

Fact $PSU_2\mathbb{C} \cong \mathrm{Rot}(S^2)$ Pf $SU_2\mathbb{C} \to SO_3\mathbb{R}$

$\quad \| $ surj w/ ker $\pm I$.

$SU_2\mathbb{C}/\pm I$

Disc $D = D_1(0) \ni w$

$$h_w(z) \longleftrightarrow \begin{pmatrix} 1 & -w \\ -\bar{w} & 1 \end{pmatrix}$$

$\mathrm{Aut}\, D \cong PSU_{1,1}(\mathbb{C})$

$$SU_{1,1}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a,b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\}$$

Upper half plane

$$H = \{ x+iy \mid y>0 \} \xrightarrow[\cong]{t} D$$

$$z \longmapsto \frac{z-i}{-iz+1}$$

so $\mathrm{Aut}(H) = t^{-1}\mathrm{Aut}(D)\, t$

and $\frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} SU_{1,1}(\mathbb{C}) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = SL_2\mathbb{R}$

so $\mathrm{Aut}(H) \cong PSL_2(\mathbb{R})$.

Aut's of H fixing $i \longleftrightarrow SO_2\mathbb{R}$

and $SL_2\mathbb{R} = PK$ for $P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a,b,d \in \mathbb{R},\ ad=1 \right\}$

G a locally compact Hausdorff top gp acting trans on a Hausdorff space X. For some $x \in X$ set G_x = isotropy subgp of x in G. Then $X \cong G/G_x$.

e.g. $S^2 \cong SO_3(\mathbb{R})/SO_2(\mathbb{R}) \cong SL_2(\mathbb{C})/P$

$H \cong SL_2\mathbb{R}/SO_2\mathbb{R}$

Riemann Mapping Theorem

Normal Families:

Defn $U \subseteq \mathbb{C}$ open. A collection $\mathcal{F}$ of analytic fns $U \to \mathbb{C}$ is a normal family if each sequence in $\mathcal{F}$ either converges uniformly to ∞ on each compact subset of U or has a subseq which converges uniformly on each compact subset of U to an analytic fn.

A collection $\mathcal{F}$ of analytic fns $U \to \mathbb{C}$ is uniformly bounded if $\exists M \geq 0$ s.t. $|f(z)| \leq M \; \forall z \in U, f \in \mathcal{F}$.

Thm [Montel] If $\mathcal{F}$ is uniformly bounded, then it is normal.

Pf Let $\{f_n\}$ be a unif bdd sequence of fns on U. Show this has subseq that converges unif on each cpt subset of U.
First enumerate pts of U w/ rat'l coords: $z_1, z_2, \ldots$
Choose $M > 0$ s.t. $|f(z)| \leq M \; \forall z \in U, f \in \mathcal{F}$. Then $\{f_n(z_1)\}$ is a seq of cpx #s bdd in modulus by $M \Rightarrow$ has conv subseq $\{f_{11}(z_1), f_{12}(z_1), \ldots\}$. Now $\{f_{1n}(z_2)\}$ bdd so has conv subseq $\rightsquigarrow \{f_{2n}\}$. Continuing inductively get $\{f_{kn}\}$ with each seq a subseq of the preceding one, and $\{f_{kn}(z_j)\}_n$ conv for $j \leq k$. Then $\{f_{nn}\}$ is a subseq converging at every z_j.

Set $g_n = f_{nn}$. Now show $\{g_n\}$ conv unif on cpt $\subseteq U$.
For $w \in U$ choose $r > 0$ s.t. $\bar{D}_{2r}(w) \subseteq U$. For $z \in \bar{D}_r(w)$

$$\bar{D}_r(z) \subseteq \bar{D}_{2r}(w) \subseteq U.$$

By Cauchy's estimates, each $f \in \mathcal{F}$ satisfies $|f'(z)| \leq \frac{M}{r}$.

For $z, z' \in \bar{D}_r(w)$, have $|f(z) - f(z')| = \left|\int_z^{z'} f'(\lambda)\,d\lambda\right| \leq \underbrace{\frac{M}{r}|z-z'|}$

independent of $f \in \mathcal{F}$ so hold $\forall g_n$.

Given $\varepsilon > 0$ choose $\delta = \frac{r\varepsilon}{3M}$. If $z \in D_r(w)$, take z_j w/ rational coeffs s.t. $|z - z_j| < \delta$.

$$|g_n(z) - g_n(z_j)| < \frac{M}{r}\frac{r\varepsilon}{3M} = \frac{\varepsilon}{3}.$$

Next choose N s.t. $|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}$ for $n, m \geq N$.
(can b/c $g_n(z_j)$ conv $\Rightarrow$ Cauchy) Then

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(z_j)| + |g_n(z_j) - g_m(z)| < \varepsilon$$

if $n, m \geq N$. Hence g_n unif Cauchy on $D_r(w)$ $\Rightarrow$ unif conv on $D_r(w)$.

For $K \subseteq U$ cpt, $\{D_r(w) \mid w \in K, D_{2r}(w) \subseteq U\}$ cover K.
Take finite subcover. Since $\{g_n\}$ unif conv on these finitely many discs, unif conv on K. $\square$.

Riemann Mapping Thm

Take $U \neq \emptyset \subsetneq \mathbb{C}$ open, conn'd, s.t. every non-vanishing analytic fn on U has an analytic square root.

Fix $z_0 \in U$ and let $\mathcal{F} = \{$inj conf fns $U \xrightarrow{f} D = D_1(0)$ & $f(z_0) = 0\}$

WTS $\exists f \in \mathcal{F}$ that surjects onto D.

Lemma $\mathcal{F}$ is nonempty.

Pf Take $\lambda \in \mathbb{C} \setminus U$, set $f(z) = z - \lambda$ which is non-vanishing on U, hence has a $\sqrt{f} = g$. Since f inj, non-van, so is g.

By Open Mapping Thm, $g(U) \supseteq \bar{D}_r(w_0)$, $0 < r < \infty$.

Note: $g^2 = f$ so $f(z) = g(z)^2 = (-g(z))^2 \Rightarrow$ if $w \in \mathrm{im}(g)$ then $-w \notin \mathrm{im}(g)$ (o/w f not inj!) Hence $\bar{D}_r(-w_0) \cap g(U) = \emptyset$.

I.e. $|g(z) + w_0| > r$ $\forall z \in U$. Thus $p(z) = \dfrac{r}{g(z)+w_0}$ is inj, conformal $U \to D$. If $p(z_0) = w$, compose w/ h_w to get $h_w \circ p : U \to D$ inj, conf taking $z_0 \mapsto 0$. $\square$

Note $p(z) = \dfrac{r}{\sqrt{z-\lambda} + w_0}$

Lemma $U, z_0, \mathcal{F}$ as above. If $f \in \mathcal{F}$ and $f(U) \subsetneq D$, then $\exists g \in \mathcal{F}$ with $|g'(z_0)| > |f'(z_0)|$.

Pf Take $w \in D \setminus f(U)$. Then $h_w \circ f(z) \neq 0$ $\forall z \in U$, so $h_w \circ f$ has an analytic $\sqrt{\ } = q$. If $\lambda^2 = w$, $q^2 = h_w \circ f$ and $q(z_0) = \lambda$, then

$$q'(z_0) = \frac{h_w'(0)}{2q(z_0)} f'(z_0) = \cdots = \frac{(1-|\lambda|^4)}{2\lambda} f'(z_0).$$

Then $g = h_\lambda \circ q \in \mathcal{F}$ and

$$g'(z_0) = h_\lambda'(\lambda)\, q'(z_0) = \cdots = \frac{(1+|\lambda|^2)}{2\lambda} f'(z_0)$$

Now $0 < (1-|\lambda|)^2 = 1 - 2|\lambda| + |\lambda|^2 \Rightarrow 2|\lambda| < 1 + |\lambda|^2$.

Also $|f'(z_0)| > 0$, so $|g'(z_0)| > |f'(z_0)|$. $\square$

Thm For U as above, there is a conf equiv $U \to D$.

Pf For $z_0 \in U$, $\mathcal{F}$ as before, know $\mathcal{F} \neq \emptyset$. Set $m = \sup\{|f'(z_0)| \mid f \in \mathcal{F}\}$

By prev lemma, existence of $h \in \mathcal{F}$ with $|h'(z_0)| = m$ implies $h(U) = D$.

Choose seq $\{f_n\}$ in $\mathcal{F}$ s.t. $\lim_{n\to\infty} |f_n'(z_0)| = m$.

Since $\mathcal{F}$ is unif bdd (by 1) it is normal, whence there is a subseq of $\{f_n\}$ converging unif on cpt $K \subseteq U$ to h.
Get $|h'(z_0)| = m$ by Cauchy's Estimates.*
Since $m \neq 0$, h is not const. Inj of $f_n \Rightarrow$ inj of h.**
Since $f_n(z_0) = 0 \ \forall n$, $h(z_0) = 0$. Thus $h \in \mathcal{F} \Rightarrow h$ conf equiv $U \longrightarrow D$. $\square$

<u>Cor</u> For U as above, U is simply conn'd.

Every $U \subsetneq \mathbb{C}$ open, simply conn'd is conf equiv to D.

ELLIPTIC FUNCTIONS (WEEK 12)

Elliptic functions are doubly periodic meromorphic functions. By *doubly periodic*, we mean that there are $\omega_1, \omega_2 \in \mathbb{C}^\times$ such that $f(z+\omega_1) = f(z) = f(z+\omega_2)$ for all $z \in \mathbb{C}$. If we assume that $\omega_2/\omega_1 \notin \mathbb{R}$, then the set $L = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ is a *lattice* in $\mathbb{C}$: a rank 2 free Abelian subgroup of $(\mathbb{C}, +)$. Let $\mathbb{C}/L$ denote the corresponding quotient group. Topologically, $\mathbb{C}/L$ is a torus, and with its complex structure it is an *elliptic curve*.[1] If f is an elliptic function with period lattice L, then it extends across the quotient map $\mathbb{C} \to \mathbb{C}/L$ to become a function on the elliptic curve $\mathbb{C}/L$. One way to understand a geometric object is by its functions, whence the importance of elliptic functions.

These notes will closely follow the development of elliptic functions in Chapter 7 of Lars Ahlfors' classic text, *Complex Analysis*; some of the later portions draw from notes by Jerry Shurman.

1. Singly periodic functions

We should walk before we run, so let's first consider *singly periodic functions*, *i.e.*, meromorphic functions f for which there exists $\omega \in \mathbb{C}$ such that $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$. We have seen examples before: the exponential function has period $2\pi i$, and $\sin$ and $\cos$ have period 2π.

Fix $\omega \in \mathbb{C}^\times$ and suppose $\Omega \subseteq \mathbb{C}$ is an open set which is closed under addition and subtraction of ω: if $z \in \Omega$, then $z \pm \omega \in \Omega$. It follows by induction that $\Omega = \Omega + \mathbb{Z}\omega$. Examples of such regions include $\mathbb{C}$ and an "open strip" parallel to ω. To better describe this open strip, transform it by dividing by ω. This has the effect of scaling by $1/|\omega|$ and rotating so that the strip is now parallel to the real axis. Thus the strip is determined by real numbers $a < b$ such that $a < \operatorname{Im}(2\pi z/\omega) < b$ for all z in the strip. (The 2π is a convenient normalization factor, as we shall shortly see.)

The function $z \mapsto \zeta = e^{2\pi i z/\omega}$ is ω-periodic. If we plug Ω into it, we get an open set in the ζ-plane. If $\Omega = \mathbb{C}$, the result is $\mathbb{C}^\times$. If Ω is the strip given by $a < \operatorname{Im}(2\pi z/\omega) < b$, the result is the annulus $e^{-b} < |\zeta| < e^{-a}$. (This follows because $e^{2\pi i z/\omega} = e^{-\operatorname{Im}(2\pi z/\omega}e^{i\operatorname{Re}(2\pi z/\omega)}$.)

Proposition 1.1. *Suppose that f is meromorphic and ω-periodic on Ω. Then there exists a unique function F on $\Omega' = e^{2\pi i\Omega/\omega}$ such that*

$$f(z) = F(e^{2\pi i z/\omega}). \tag{1}$$

Proof. To determine $F(\zeta)$, first note that $\zeta = e^{2\pi i z/\omega}$ for some $z \in \Omega$, and that z is unique up to addition of an integer-multiple of ω. Since f is ω-periodic, the formula $F(\zeta) = f(z)$ is well-defined, and it is clearly meromorphic in Ω'. Uniqueness follows from noting that when F is meromorphic on Ω', equation (1) defines a function f meromorphic on Ω with period ω. $\square$

Now suppose that Ω' contains an annulus $r < |\zeta| < R$ on which F has is analytic. On this annulus, F has a Laurent series

$$F(\zeta) = \sum_{n=-\infty}^{\infty} c_n \zeta^n,$$

whence

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z/\omega}.$$

[1] Curves are one-dimensional and tori are two-dimensional. What gives? The 'curve' in 'elliptic curve' indicates a single *complex* dimension.

This is the *complex Fourier series* for f in the strip $-\log(R) < \mathrm{Im}(2\pi z/\omega) < -\log r$.

By old formulae, we know that for $r < s < R$,

$$c_n = \frac{1}{2\pi i}\int_{|\zeta|=s}\frac{F(\zeta)}{\zeta^{n+1}}\,d\zeta,$$

which, by change of variables, is equivalent to

$$c_n = \frac{1}{\omega}\int_d^{d+\omega} f(z)e^{-2\pi i n z/\omega}\,dz.$$

Here d is an arbitrary point in the strip corresponding to the annulus, and the integration is along any path from d to $d+\omega$ which remains in the strip. (You will verify the final details of this in your homework.) We have thus proven the following result.

Theorem 1.2. *Suppose f is meromorphic and ω-periodic on an open set $\Omega \subseteq \mathbb{C}$ and is analytic on the strip given by $a < \mathrm{Im}(2\pi z/\omega) < b$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z/\omega}$$

for z in the strip, and

$$c_n = \frac{1}{\omega}\int_d^{d+\omega} f(z)e^{-2\pi i n z/\omega}\,dz$$

for d in the strip and the integration along any path from d to $d+\omega$ in the strip. If f is analytic on $\mathbb{C}$, then the Fourier series is valid on $\mathbb{C}$ as well. $\square$

2. Doubly periodic functions

An *elliptic function* is a meromorphic function on the plane with two periods, $\omega_1, \omega_2 \in \mathbb{C}$ such that $\omega_2/\omega_1 \notin \mathbb{R}$. The significance of the final condition is that one of the periods is not a real scaling of the other. This has the effect of making $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ a rank 2 free Abelian group inside $(\mathbb{C}, +)$, as we shall currently show.

2.1. **The period lattice.** For the moment, forget the condition on ω_2/ω_1 and just suppose that $f(z+\omega_1) = f(z) = f(z+\omega_2)$ for all $z \in \mathbb{C}$. Let $M := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ denote the *period module* of f.

Proposition 2.1. *If f is not constant with periods $\omega_1, \omega_2 \in \mathbb{C}^\times$, then $M = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is discrete.*

Proof. Since $f(\omega) = f(0)$ for all $\omega \in M$, the existence of an accumulation point in M would imply that f is constant (by the Identity Theorem). $\square$

Theorem 2.2. *A discrete subgroup A of $(\mathbb{C}, +)$ is either*
(0) rank 0: $A = \{0\}$,
(1) rank 1: $A = \mathbb{Z}\omega$ for some $\omega \in \mathbb{C}^\times$, or
(2) rank 2: $A = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for some $\omega_1, \omega_2 \in \mathbb{C}^\times$ with $\omega_2/\omega_1 \notin \mathbb{R}$.

Proof. We may assume that $A \neq \{0\}$. Take $r > 0$ such that $\overline{D}_r(0) \cap A$ contains more than just 0. Since $\overline{D}_r(0)$ is compact and A is discrete, the intersection contains only finitely many points. Choose one with minimum nonzero modulus and call it ω_1. (You can check that there are always exactly two, four, or six points in A closest to 0.) Then $\mathbb{Z}\omega_1 \subseteq A$.

If $A = \mathbb{Z}\omega_1$, we are in case (1) and done. Suppose there exists $\omega \in A \setminus \mathbb{Z}\omega_1$. Among all such ω, there exists one, ω_2, of smallest modulus. Suppose for contradiction that $\omega_2/\omega_1 \in \mathbb{R}$. Then we could find an integer n such that $n < \omega_2/\omega_1 < n+1$. It would follow that $|n\omega_1 - \omega_2| < |\omega_1|$, a contradiction.

We now aim to show that $A = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. We claim that every $z \in \mathbb{C}$ may be written as $z = \lambda_1\omega_1 + \lambda_2\omega_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$. To see this, we attempt to solve the equations

$$\begin{aligned} z &= \lambda_1\omega_1 + \lambda_2\omega_2 \\ \overline{z} &= \lambda_1\overline{\omega}_1 + \lambda_2\overline{\omega}_2. \end{aligned}$$

The determinant $\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1 \neq 0$ (otherwise ω_2/ω_1 is real) and thus the system has a unique solution $(\lambda_1, \lambda_2) \in \mathbb{C}^2$. But clearly $(\overline{\lambda}_1, \overline{\lambda}_2)$ is a solution as well, so $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, as desired.

Now choose integers m_1, m_2 such that $|\lambda_1 - m_1| \leq 1/2$ and $|\lambda_2 - m_2| \leq 1/2$. If $z \in A$, then $z' = z - m_1\omega_1 - m_2\omega_2 \in A$ as well. Thus $|z'| < \frac{1}{2}|\omega_1| + \frac{1}{2}|\omega_2| \leq |\omega_2|$. (The first inequality is strict since ω_2 is not a real multiple of ω_1.) Since ω_2 has minimal modulus in $A \setminus \mathbb{Z}\omega_1$, we learn that $z' \in \mathbb{Z}\omega_1$, say $z' = n\omega_1$. Thus $z = (m_1 + n)\omega_1 + m_2\omega_2 \in \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and we conclude that $A = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. □

2.2. **The modular group.** From now on, we assume that the period lattice has rank 2. Any pair (ω_1, ω_2) such that $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is called a *basis* of L (and necessarily satisfies $\omega_2/\omega_1 \notin \mathbb{R}$).

Suppose that (ω_1', ω_2') is another basis of L. Then there exist $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2. \end{aligned}$$

In matrix form, this is

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

The same relation is valid for the complex conjugates, so

$$\begin{pmatrix} \omega_1' & \overline{\omega}_1' \\ \omega_2' & \overline{\omega}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2 \end{pmatrix}.$$

Since (ω_1', ω_2') is also a basis, there are also integers a', b', c', d' such that

$$\begin{pmatrix} \omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' & \overline{\omega}_1' \\ \omega_2' & \overline{\omega}_2' \end{pmatrix}.$$

Substituting, we get

$$\begin{pmatrix} \omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2 \end{pmatrix}.$$

We know that $\det \begin{pmatrix} \omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2 \end{pmatrix} \neq 0$ (since $\omega_2/\omega_1 \notin \mathbb{R}$), and thus we may multiply on the right by $\begin{pmatrix} \omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2 \end{pmatrix}^{-1}$ to get

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the integer matrices are inverses of each other, and their determinants multiply to give 1. Since both determinants are integers, we see that $ad - bc$ and $a'd' - b'c'$ are ± 1. Let $\mathrm{GL}_2(\mathbb{Z}) := \{m \in M_{2\times 2}(\mathbb{Z}) \mid \det m = \pm 1\}$ denote the General Linear group of 2×2 invertible integer matrices. We have proven the following result.

Theorem 2.3. *Suppose $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$ with ordered basis (ω_1, ω_2). Then the set of all ordered bases of L is the $\mathrm{GL}_2(\mathbb{Z})$-orbit of (ω_1, ω_2),* i.e., *the set of (ω_1', ω_2') such that*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

for some integers a, b, c, d with $ad - bc = \pm 1$. □

The group $\mathrm{GL}_2(\mathbb{Z})$ is called the *modular group*. That term, though, is also sometimes used for $\mathrm{SL}_2(\mathbb{Z})$, the 2×2 integer matrices with determinant 1. This latter group can be thought of as the transformations that change basis in an orientation-preserving fashion.

2.3. **The canonical basis.** We now single out a nearly unique basis called the *canonical basis* of a lattice L.

Theorem 2.4. *Given a lattice L, there exists a basis (ω_1, ω_2) such that $\tau = \omega_2/\omega_1$ satisfies the following conditions:*

(i) $\mathrm{Im}(\tau) > 0$,
(ii) $-1/2 < \mathrm{Re}(\tau) \leq 1/2$,
(iii) $|\tau| \geq 1$, *and*
(iv) if $|\tau| = 1$, *then* $\mathrm{Re}(\tau) \geq 0$.

The ratio τ is uniquely determined by these conditions, and there is a choice of two, four, or six corresponding ordered bases.

Proof. Choose ω_1 and ω_2 as in the proof of Theorem 2.2. Then $|\omega_1| \leq |\omega_2| \leq |\omega_1 \pm \omega_2|$. In terms of $\tau = \omega_2/\omega_1$, the first inequality becomes $|\tau| \geq 1$. Dividing the second inequality by $|\omega_1|$ we get $|\tau| \leq |1 \pm \tau|$. Squaring and expanding by real and imaginary parts gives

$$\mathrm{Re}(\tau)^2 + \mathrm{Im}(\tau)^2 \leq (1 \pm \mathrm{Re}(\tau))^2 + \mathrm{Im}(\tau^2).$$

Canceling, expanding, and rearranging gives

$$0 \leq 1 \pm 2\,\mathrm{Re}(\tau),$$

i.e., $|\,\mathrm{Re}(\tau)| \leq 1/2$.

If $\mathrm{Im}(\tau) < 0$, replace (ω_1, ω_2) by $(-\omega_1, \omega_2)$, making $\mathrm{Im}(\tau) > 0$ without changing $\mathrm{Re}(\tau)$. If $\mathrm{Re}(\tau) = -1/2$, replace the basis by $(\omega_1, \omega_1 + \omega_2)$, and if $|\tau| = 1$ with $\mathrm{Re}(\tau) < 0$, replace it by $(-\omega_2, \omega_1)$. After these changes, all the conditions are satisfied. Uniqueness will be handled in Theorem 2.6.

There are always at least two bases corresponding to $\tau = \omega_2/\omega_1$, namely (ω_1, ω_2) and $(-\omega_1, -\omega_2)$. We handle the exceptional cases of 4 and 6 bases after the proof of 2.6. $\square$

Definition 2.5. The collection of τ described by Theorem 2.4 is called the *fundamental region* of the unimodular group.

The unimodular group $\mathrm{GL}_2(\mathbb{Z})$ acts on bases (ω_1, ω_2) via matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a\omega_2 + b\omega_1 \\ c\omega_2 + d\omega_1. \end{pmatrix}$$

(We have swapped the usual order of ω_1 and ω_2 so as to more closely mirror $\tau = \frac{\omega_2}{\omega_1}$.) As such, it acts on the quotient $\tau = \omega_2/\omega_1$ via a linear fractional transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1} = \frac{a\tau + b}{c\tau + d}$$

Theorem 2.6. *If τ and τ' are in the fundamental region and $\tau' = (a\tau + b)/(c\tau + d)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, then $\tau = \tau'$.*

Proof. Suppose that $\tau' = (a\tau + b)/(c\tau + d)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. Then

$$\mathrm{Im}(\tau') = \frac{\pm\,\mathrm{Im}(\tau)}{|c\tau + d|^2}$$

with sign matching that of $ad - bc = \pm 1$. If τ and τ' are in the fundamental region, then the sign must be positive, so $ad - bc = 1$. Without loss of generality, $\mathrm{Im}(\tau') \geq \mathrm{Im}(\tau)$, so $|c\tau + d| \leq 1$.

If $c = 0$, then $d = \pm 1$ or 0. Since $ad - bc = 1$, we have $ad = 1$, so either $a = d = 1$ or $a = d = -1$. Then $\tau' = \tau \pm b$, whence $|b| = |\operatorname{Re}(\tau') - \operatorname{Re}(\tau)| < 1$. Therefore $b = 0$ and $\tau = \tau'$.

We leave the $c \neq 0$ case as a moral exercise for the reader. The arguments are somewhat intricate, but unsurprising. ☐

Finally, we note that τ corresponds to bases other than (ω_1, ω_2) and $(-\omega_1, -\omega_2)$ if and only if τ is a fixed point of some unimodular transformation. This only happens for $\tau = i$ (which is a fixed point of $-1/\tau$) and $\tau = e^{\pi i/3}$ (which is a fixed point of $-(\tau+1)/\tau$ and $-1/(\tau+1)$.) We leave it to the reader to check that these are the only possibilities.

2.4. **General properties of elliptic functions.** Let f be a meromorphic function on $\mathbb{C}$ with period lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ of rank 2. (We do not assume that (ω_1, ω_2) is a canonical basis, nor do we assume that L comprises all periods of f.)

Some notation to ease our upcoming work: write $z_1 \equiv z_2 \pmod{L}$ if $z_1 - z_2 \in L$. For $a \in \mathbb{C}$, write P_a for the "half open" parallelogram with vertices $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ that includes the line segments $[a, a+\omega_1]$ and $[a+\omega_1, a+\omega_1+\omega_2)$ but does not include the other two sides. Then every point in $\mathbb{C}/L$ (the set of equivalence classes modulo L) contains a unique representative in P_a.

Theorem 2.7. *If f is an elliptic function without poles, then f is constant.*

Proof. If f is analytic on P_a, then it is bounded on the closure of P_a, and hence bounded and analytic on $\mathbb{C}$. By Liouville's theorem, f is constant. ☐

Proposition 2.8. *An elliptic function has finitely many poles in P_a.*

Proof. Poles always form a discrete set, and P_a is bounded. ☐

Theorem 2.9. *The sum of the residues of an elliptic function at poles in a parallelogram P_a is zero.*

Proof. We may perturb a so that none of the poles lie on ∂P_a. Then, by the residue theorem, the sum of the residues at poles in P_a equals

$$\frac{1}{2\pi i} \int_{\partial P_a} f(z)\, dz.$$

Since f has periods ω_1 and ω_2, the line integrals along opposite sides cancel, and we get that the sum of the residues is 0. ☐

Corollary 2.10. *No elliptic function has a single simple pole (and no other poles) in some P_a.*

Proof. A simple pole has a nonzero residue, and the sum of the residues is zero. ☐

Theorem 2.11. *A nonzero elliptic function has equally many poles and zeros in any P_a (where poles and zeroes are counted with multiplicity).*

Proof. Fix a nonzero elliptic function f. In the proof of Theorem 4.4.7 we saw that the *logarithmic derivative* f'/f has the zeros and poles of f as simple poles, with residues equal to their (signed) multiplicities. Since f'/f is also elliptic, the result follows from Theorem 2.9. ☐

Note that for any constant $c \in \mathbb{C}$, $f(z) - c$ has the same poles as $f(z)$. It follows that all values are assumed the same number of times by f.

Definition 2.12. The number of incongruent (mod L) roots of the equations $f(z) = c$ is called the *order* of the elliptic function.

Theorem 2.13. *The zeros* $a_1, \dots, a_n$ *and poles* $b_1, \dots, b_n$ *of an elliptic function satisfy* $a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \pmod{L}$.

Proof. Choose $a \in \mathbb{C}$ such that none of the zeros or poles are on ∂P_a. Also choose zeros and poles inside of P_a. By calculus of residues,

$$\frac{1}{2\pi i} \int_{\partial P_a} \frac{z f'(z)}{f(z)}\, dz = a_1 + \cdots + a_n - b_1 - \cdots - b_n.$$

(Check this!) It remains to prove that the left-hand side is an element of $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The portion of the integral contributed by the sides $[a, a + \omega_1]$ and $[a + \omega_2, a + \omega_1 + \omega_2]$ is

$$\frac{1}{2\pi i} \left(\int_a^{a+\omega_1} - \int_{a+\omega_2}^{a+\omega_1+\omega_2} \right) \frac{z f'(z)}{f(z)}\, dz = -\frac{\omega_2}{2\pi i} \int_a^{a+\omega_1} \frac{f'(z)}{f(z)}\, dz.$$

(Check this!) As z varies in $[a, a + \omega_1]$, the values $f(z)$ describe a closed curve in the plane; call this curve γ. Then the right-hand side of the above expression is manifestly $-\omega_2 \operatorname{Ind}_\gamma(0)$, which is an integer multiple of ω_2. A similar argument applies to the other pair of opposite sides. We conclude that

$$a_1 + \cdots + a_n - b_1 - \cdots - b_n = m\omega_1 + n\omega_2$$

for some integers m, n, as desired. ∎

ELLIPTIC FUNCTIONS (WEEK 13)

3. The Weierstrass $\wp$-function

Following Weierstrass, we now create our first example of an elliptic function. The simplest examples will have order 2 (the smallest possible order) and necessarily have either a single double pole with residue zero, or two simple poles with opposite residues. Our example will have a double pole with residue zero.

We begin with a list of *desiderata* and their necessary implications. We want $\wp = \wp(\ ;\omega_1, \omega_2)$ to be elliptic with a double pole at 0 and periods $\omega_1, \omega_2 \in \mathbb{C}^\times$ such that $\omega_2/\omega_1 \notin \mathbb{R}$. Thus the leading term in the Laurent series of $\wp$ may as well be z^{-2}. Now $\wp(z) - \wp(-z)$ has the same periods and no singularity, hence is constant. Furthermore $\wp(\omega_1/2) - \wp(-\omega_1/2 = 0)$ by ω_1-periodicity of $\wp$, so $\wp(z) - \wp(-z) = 0$ for all z. We conclude that $\wp$ is an even function.

Addition of a constant is inconsequential, so let's demand that $\wp$'s constant term is 0. Thus we are on the hunt for a function of the form

$$\wp(z) = z^{-2} + a_1 z^2 + a_2 z^4 + a_3 z^6 + \cdots$$

with periods ω_1, ω_2.

Let $L = \mathbb{Z}\omega_1 + \omega_2$. We aim to show that

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L \smallsetminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

This is a reasonable formula to guess: we get poles of order 2 at all points in the period lattice, and $1/\omega^2$ is subtracted (making the summands roughly z/ω^3) to guarantee uniform convergence on compact sets. It is not obviously L-periodic (because of the $-1/\omega^2$ term), but you can show that

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z-\omega)^3}.$$

This function is clearly L-periodic, and you will combine this with evenness of $\wp$ to prove that $\wp$ has periods ω_1, ω_2 in a homework problem.

Having built up our desired properties, we will make one final definition and then state an omnibus theorem summarizing the properties of $\wp$.

Definition 3.1. The k-th *Eisenstein series* of a lattice L is

$$G_k = G_k(L) = \sum_{\omega \in L \smallsetminus \{0\}} \frac{1}{\omega^k}.$$

Remark 3.2. If k is odd, $G_k = 0$.

Theorem 3.3. *Let $\wp$ be the Weierstrass function with respect to a lattice L.*

(a) The Laurent expansion of $\wp$, valid for $0 < |z| < \min\{|\omega| \mid 0 \neq \omega \in L\}$, is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n}.$$

(b) The functions $\wp$ and $\wp'$ satisfy the differential equation

(1) $$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$.

(c) If $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, let $\omega_3 = \omega_1 + \omega_2$ and set $e_i = \wp(\omega_i/2)$ for $i = 1, 2, 3$. Then (1) is equivalent to

(2) $$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

and the e_i are distinct.

Some interpretation of (b) and (c) is in order. By (1), we know that the pair $(\wp(z), \wp'(z))$ satisfies the equation

$$y^2 = 4x^3 - g_2x - g_3$$

for $z \in \mathbb{C}$. It is in fact the case that the assignment

$$\begin{aligned} \mathbb{C}/L &\longrightarrow \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2x - g_3\}^* \\ z + L &\longmapsto (\wp(z), \wp'(z)) \end{aligned}$$

is a bijection (where the * indicates adding a point at ∞, and $L \mapsto \infty$). The object on the right is an algebraic geometer's notion of an elliptic curve, and this bijection explains the duplication of terminology.

Furthermore, (2) says that the right-hand side has roots e_1, e_2, e_3, giving the equivalent equation

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3).$$

Since the e_i are distinct, we call this equation *nonsingular*.

Proof of Theorem 3.3 (sketch). For (a), note that for $|z| < |\omega|$, the summand

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{(1 - z/\omega)^2} - 1\right) = \frac{1}{\omega^2}\sum_{n=1}^{\infty}(n+1)\frac{z^n}{\omega^n}$$

where the last equality follows from squaring the geometric series. Thus the summand is equal to $2z/\omega^3 + 3z^2/\omega^4 + \cdots$. Reordering the summations gives the desired identity.

For (b), compare the Laurent series in question. We have

$$\wp(z) = \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + O(z^6)$$

and

$$\wp'(z) = -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + O(z^5).$$

By some algebra, both $(\wp'(z))^2$ and $4\wp(z)^3 - g_2\wp(z) - g_3$ are of the form

$$\frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + O(z^2).$$

It follows that $(\wp'(z))^2 - (4\wp(z)^3 - g_2\wp(z) - g_3)$ is analytic and elliptic, hence constant. Since the difference is $O(z^2)$, it is also equal to 0.

For (c), recall that $\wp'$ is odd, and suppose that z is a point of order 2 in $\mathbb{C}/L$. Then $z \equiv -z \pmod{L}$, and $\wp'(z) = \wp'(-z) = -\wp'(z)$, whence $\wp'(z) = 0$. The order 2 points in $\mathbb{C}/L$ are exactly $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$, and (2) follows from (b). It remains to show that the e_i are distinct, but this follows because each is a double value of $\wp$ (since $\wp' = 0$ at the corresponding z-values) and $\wp$ has order 2. $\square$

4. THE DISCRIMINANT AND j-FUNCTION

For $\tau \in \mathfrak{h} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, set $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$, the lattice with basis $(\tau, 1)$. We can turn the Eisenstein series into functions of the variable $\tau \in \mathfrak{h}$ by setting

$$G_k(\tau) = G_k(L_\tau).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned} G_k(\tau) &= G_k(L_\tau) \\ &= G_k(\gamma L_\tau) \\ &= G_k(\mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d)) \\ &= G_k((c\tau + d)\left(\frac{a\tau + b}{c\tau + d}\mathbb{Z} + \mathbb{Z}\right)) \\ &= (c\tau + d)^{-k} G_k(L_{\gamma\tau}) \\ &= (c\tau + d)^{-k} G_k(\gamma\tau). \end{aligned}$$

Here the second to last equality follows from the elementary observation that $G_k(mL) = m^{-k}G_k(L)$. Summarizing, we get

$$G_k(\gamma\tau) = (c\tau + d)^k G_k(\tau)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{h}$.

We now define the *discriminant function*

$$\begin{aligned} \Delta : \mathfrak{h} &\longrightarrow \mathbb{C} \\ \tau &\longmapsto g_2(\tau)^3 - 27g_3(\tau)^2 \end{aligned}$$

which satisfies the transformation law

$$\Delta(\gamma\tau) = (c\tau + d)^{12}\Delta(\tau).$$

This permits the definition of *Klein's j-function*,

$$\begin{aligned} j : \mathfrak{h} &\longrightarrow \mathbb{C} \\ \tau &\longmapsto 1728\frac{g_2(\tau)^3}{\Delta(t)} \end{aligned}$$

which is $\mathrm{SL}_2(\mathbb{Z})$-equivariant:

$$j(\gamma\tau) = j(\tau).$$

In fact, j is a holomorphic isomorphism between $X = \mathfrak{h}^*/\mathrm{SL}_2(\mathbb{Z})$ and the Riemann sphere (where $j(\infty) = \infty$). The space X is the *moduli space* of elliptic curves, and j specifies its topology and complex structure.

5. FIELDS OF MEROMORPHIC FUNCTIONS

A *Riemann surface* is a space in which every point admits an open neighborhood conformally equivalent to an open subset of $\mathbb{C}$. We have been working with three primary examples: open subsetes of $\mathbb{C}$, S^2, and $\mathbb{C}/L$. A more exotic example is the modular surface $\mathfrak{h}^*/\mathrm{SL}_2(\mathbb{Z})$.

One way to probe a Riemann surface is to understand its functions. Presently, we will concern ourselves with meromorphic functions on a Riemann surface X. These are the analytic functions $X \to S^2$ which are not constant with value ∞. As such, a function like $z \mapsto e^z/z$ is meromorphic on $\mathbb{C}$ but not on S^2. (It has an essential singularity at ∞.) We may pointwise add, subtract, multiply, and divide meromorphic functions on X (with some care, *i.e.*, limits, in cases like $0 \cdot \infty$), and this gives the set $K(X)$ of meromorphic functions on X the structure of a field. In general, functions on compact Riemann surfaces tend to be much simpler than on non-compact surfaces, and we will currently describe the meromorphic functions on S^2 and $\mathbb{C}/L$.

5.1. **Functions on the Riemann sphere.** Meromorphic functions on $S^2 = \mathbb{C} \cup \{\infty\}$ are particularly nice. First suppose that $f : S^2 \to S^2$ restricts to a function $f : \mathbb{C} \to \mathbb{C}$. This is our old notion of an entire function with the additional restriction that f has a nonessential singularity at ∞. By methods similar to one of Exam 2's problems, we can show that such functions are polynomial.

Now suppose that $f : S^2 \to S^2$ is analytic and takes the value ∞ (*i.e.* has poles as a function on $\mathbb{C}$) at $z_1, \dots, z_n \in \mathbb{C}$. If these poles have orders $k_1, \dots, k_n$, respectively, then the function

$$g : S^2 \longrightarrow S^2$$
$$z \longmapsto g(z) \prod_{i=1}^{n} (z - z_i)^{k_i}$$

is entire when restricted to $\mathbb{C}$. Thus g is a polynomial function, and

$$f(z) = \frac{g(z)}{\prod_{i=1}^{n} (z - z_i)^{k_i}}.$$

This proves the following theorem.

Theorem 5.1. *The field of meromorphic functions on the Riemann sphere equals the field of rational functions in a single variable,* i.e.,

$$K(S^2) = \mathbb{C}(z) = \{p(z)/q(z) \mid p, q \textit{ polynomials with coefficients in } \mathbb{C}, \, q \neq 0\}.$$

5.2. **Functions on elliptic curves.** Fix a lattice $L = \mathbb{Z}\omega_1 + \omega_2$ and let $\wp = \wp(\ ; L)$ be the associated Weierstrass $\wp$-function. Miraculously, we only need to know $\wp$ in order to know all of the meromorphic functions on $\mathbb{C}/L$.

Theorem 5.2. *The field $K(\mathbb{C}/L)$ consists of rational functions in $\wp$ and $\wp'$,* i.e.,

$$K(\mathbb{C}/L) = \mathbb{C}(\wp, \wp') = \left\{ \frac{f(\wp, \wp')}{g(\wp, \wp')} \,\middle|\, f, g \textit{ polynomials in two variable with coefficients in } \mathbb{C}, \, g \neq 0 \right\}.$$

Furthermore,

$$\mathbb{C}(\wp, \wp') \cong \mathbb{C}(x, y)/(y^2 = 4x^3 - g_2 x - g_3) = \mathbb{C}(x)(\sqrt{4x^3 - g_2 x - g_3})$$

the field of rational functions in two variables x, y subject to the relation $y^2 = 4x^3 - g_2 x - g_3$ where $g_i = g_i(L)$.

First note that $\mathbb{C}(\wp, \wp')$ is clearly a subfield of $K(\mathbb{C}/L)$, and the relation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

of Theorem 3.3 implies the final isomorphism. What is surprising here is that *every* meromorphic function on $\mathbb{C}/L$ can be expressed in such a fashion, and that is what we will concern ourselves with in the following sketch.

Proof Sketch. We begin with a reduction step that will allow us to only consider the even elliptic functions. Suppose f is meromorphic on $\mathbb{C}/L$ and let

$$f_1(z) = \frac{f(z) + f(-z)}{2}, \quad f_2(z) = \frac{f(z) - f(-z)}{2\wp'(z)}.$$

Since $\wp'$ is odd, both of these functions are even, and $f = f_1 + \wp' \cdot f_2$. As such, it suffices to prove that the field of even meromorphic functions on $\mathbb{C}/L$ is $\mathbb{C}(\wp)$.

Suppose $f : \mathbb{C}/L \to S^2$ is meromorphic and even. Our strategy is to produce an even meromorphic function φ on $\mathbb{C}/L$ which is rational in $\wp$ and has the same order of vanishing as f at all points. It will then follow that f/φ is analytic and elliptic, and thus is constant, from which we conclude that $f = c\varphi$ is rational in $\wp$ as well.

Let $\nu_0(f)$ denote the order of vanishing of f near 0. The Laurent series of f about 0 takes the form

$$f(z) = \sum_{n \geq \nu_0(f)} a_n z^n$$

where all powers of n are even and thus $\nu_0(f)$ is even. Near $\omega_1/2$, we have a similar expansion

$$f(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n (z - \omega_1/2)^n.$$

Define $g(z) = f(z + \omega/2)$, which is also meromorphic on $\mathbb{C}/L$. This function is also even since

$$g(-z) = f(-z + \omega_1/2) = f(-z - \omega_1/2 + \omega_1) = f(-z - \omega_1/2) = g(z).$$

Thus $\nu_0(g)$ is even as well. Additionally, the Laurent expansion of g about 0 is

$$g(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n z^n$$

so $\nu_{\omega_1/2}(f)$ is even as well. Via similar arguments, $\nu_{\omega_2/2}(f)$ and $\nu_{(\omega_1+\omega_2)/2}(f)$ are even as well.

Let $\{\pm z_1, \ldots, \pm z_n\}$ be the set of congruence classes of zeros or poles of f not of the form $(\varepsilon_1\omega_1 + \varepsilon_2\omega_2)/2$ for $\varepsilon_i = 0$ or 1. (The latter classes are precisely those z for which $z = -z$ in $\mathbb{C}/L$.) Let $(\mathbb{C}/L)[2]$ denote these 2-torsion points. Define φ by the formula

$$\varphi(z) = \prod_{i=1}^{n} (\wp(z) - \wp(z_i))^{\nu_{z_i}(f)} \prod_{w \in (\mathbb{C}/L)[2]} (\wp(z) - \wp(w))^{\nu_w(f)/2}.$$

(We have seen that $\nu_w(f)$ is even, and this value is 0 when w is not a zero or pole of f, in which case the term does not contribute to the product.) Clearly, this is a rational function in $\wp$. Furthermore, φ has the same order of vanishing as f everywhere since $\wp$ takes the values in W to order 2 and takes all other values to order 1. Thus we have produced the desired φ and $f = c\varphi$ is rational in $\wp$ as well, completing the argument. $\square$