MATH 202: VECTOR CALCULUS
DETERMINANTS DONE HASTILY

The determinant function from linear algebra has important applications in vector calculus. Determinants were covered in Math 201 and you can review them in detail in §§3.5-3.9 of *CAES*, but we will present a brief recollection here of the aspects of the determinant most important to our study.

1. Abstract nonsense

Throughout, let F be a field and let V be an n-dimensional F-vector space, $n < \infty$. We write $\mathrm{End}_F(V) = \mathrm{End}(V)$ for the collection of F-linear *endomorphisms* of V, that is, F-linear map $V \to V$. A choice of ordered basis for V allows us to identify $\mathrm{End}(V)$ with $M_n(F)$, $n \times n$ matrices with entries in F.[1] Thinking of an $n \times n$ matrix as an n-tuple of its row vectors, we may in turn identify $M_n(F)$ (and hence $\mathrm{End}(V)$) with V^n.

Defineorem 1.1. The *determinant* of an $n \times n$ matrix $A \in M_n(F)$ is the unique multilinear skew-symmetric normalized function $\det : M_n(F) \to F$.

Some comments are in order. The word *defineorem* is a portmanteau of definition and theorem, and is not standard in the mathematical literature.[2] A theorem says that there exists a unique multilinear skew-symmetric normalized function $M_n(F) \to F$, and we then define the determinant to be this function.

We should also recall the meaning of the terms invoked:

» A *multilinear* function $M_n(F) \to F$ is a function which is linear in each row.
» Such a function is *skew-symmetric* if exchanging rows multiplies the output of the function by -1.
» Such a function is *normalized* if its value on the identity matrix is 1.

Defineorem 1.2. The *determinant* of a linear transformation $L \in \mathrm{End}(V)$ is the determinant of the matrix associated with L for any choice of basis.

In particular, any choice of basis produces the same linear transformation. This is a consequence of multiplicativity of the determinant, our next theorem.

Theorem 1.3. *For all* $A, B \in M_n(F)$, $\det(AB) = \det(A)\det(B)$.

Corollary 1.4. *A matrix* $A \in M_n(F)$ *is invertible if and only if* $\det(A) \neq 0$.

In the above discussion, we privileged multilinearity of $\det$ with respect to the rows of a matrix. In fact, we could have set everything up in terms of columns instead because of the following theorem. Recall that the *transpose* of a square matrix $A = (a_{ij})$ is the matrix $A^\top := (a_{ji})$. We can think of this as reflection across the diagonal, or as swapping rows and columns.

Theorem 1.5. *For all* $A \in M_n(F)$, $\det(A^\top) = \det(A)$.

[1] If $v_1, \ldots, v_n$ is an ordered basis of V, $L \in \mathrm{End}(V)$, and $L(v_j) = \sum_{i=1}^n x_{ij} v_i$, then the associated matrix is $(x_{ij})_{i,j=1}^n$. In other words, the j-th column of the matrix is column vector representation of $L(v_j)$ in terms of the ordered basis.

[2] Though, in this author's opinion, it should be.

2. Formulas

Let Σ_n be the collection of permutations of $\{1, 2, \ldots, n\}$, *i.e.*, bijective functions $\{1, \ldots, n\} \to \{1, \ldots, n\}$. This is called the *symmetric group on n letters*. Recall that every permutation π has a *sign*, denoted $\operatorname{sgn}(\pi)$ or $(-1)^\pi$ in $\{\pm 1\}$ which is 1 if π is the composition of an even number of transpositions and is -1 if π is the composition of an odd number of transpositions.

Theorem 2.1. *For every $A = (a_{ij}) \in M_n(F)$,*

$$\det(A) = \sum_{\pi \in \Sigma_n} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

There are $n!$ terms in that expansion. Yikes! Here are some simple low-dimensional cases.

Proposition 2.2. *For all $a, b, c, d, e, f, g, h, k \in \mathbb{R}$,*

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

and

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = aek + bfg + cdh - afh - bdk - ceg.$$

A visual mnemonic for the second identity can be found on p.102 of *CAES*.

We can also compute determinants via Laplace's formula, aka expansion by minors. The *$(i, j)-$th minor* of an $n \times n$ matrix A, denoted $M_{ij}(A)$, is the determinant of the $(n-1) \times (n-1)$ matrix created by deleting the i-th row and j-th column from A. If the matrix A is understood by context, we will shorten this to M_{ij}.

Theorem 2.3 (Laplace's formula)**.** *For any $A = (a_{ij}) \in M_n(F)$ and $1 \leq k \leq n$*

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj} = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} M_{ik}.$$

The first sum is the expansion along the k-th row, while the second sum is expansion along the k-th column.

3. Volume

In this section, we specialize to the case $F = \mathbb{R}$. Let $L \in \operatorname{End}(\mathbb{R}^n)$ have associated matrix A_L with respect to the standard ordered basis $e_1, \ldots, e_n$. Let U denote the (solid) unit cube in $\mathbb{R}^n$ with vertices $\varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_n e_n$ where each $\varepsilon_i = 0$ or 1. Let us further agree that the volume of U is $\operatorname{vol}(U) = 1$. We denote the image of U under L by LU and note that this is a (possibly degenerate) parallelipiped.

Theorem 3.1. *For and $L \in \operatorname{End}(\mathbb{R}^n)$,*

$$\operatorname{vol}(LU) = |\det(L)|.$$

In fact, for any reasonable notion of volume and a "measurable" set $\mathcal{E} \subseteq \mathbb{R}^n$,

$$\operatorname{vol}(L\mathcal{E}) = |\det(L)| \operatorname{vol}(\mathcal{E}),$$

but we won't get into the details of this assertion in this class.

4. ORIENTATION

We just recalled that the absolute value of the determinant of a linear mapping is a scaling factor for how volume transforms under the mapping. What does the sign of the determinant tell us?

Recall that a square matrix is invertible if and only if its determinant is nonzero if and only if its constituent column vectors are linearly independent. So the determinant is 0 precisely when there is some linear dependence between the columns. If the columns form a basis of $\mathbb{R}^n$, then the determinant of the basis will either be positive or negative, and we then lend these terms (positive or negative) to the ordered basis.

Since $\det$ is continuous when considered as a function $M_n(\mathbb{R}) = \mathbb{R}^{n^2} \to \mathbb{R}$ (you'll prove this in your homework), the intermediate value theorem tells us that a continuous path of ordered bases never switches signs. In fact, a basis is positive if and only if there is a continuous path of bases to the standard basis $e_1, \ldots, e_n$, but we will not prove this here. (If the idea of paths in spaces of bases sounds exciting, make sure to ask your topology instructor about Stiefel manifolds.)